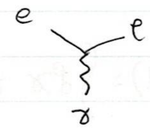
外場による散乱

適切な外場 $A_e^\mu(x)$ による電子の散乱を考える。

電磁場 $A^\mu(x)$ を $A^\mu(x) + A_e^\mu(x)$ と置き換える。
量子場　外場

全て Free state だと



は発散する。

このとき、作用は

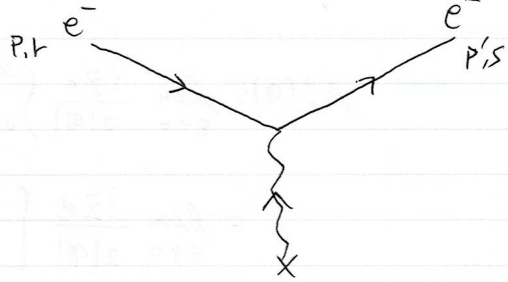
$$S = \sum_{n=0}^{\infty} \frac{(ie)^n}{n!} \int\cdots\int d^4x_1 \cdots d^4x_n \, T\left\{ N\left[\bar{\psi}(A + A_e)\psi\right]_{x_1} \times \cdots \times N\left[\bar{\psi}(A + A_e)\psi\right]_{x_n} \right\}$$

標的粒子が非常に重い等の場合、

外場は時間に依存しない ⇔ 定常状態と考えられる。

よって、

$$A_e^\mu(x) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot x} \cdot A_e^\mu(q)$$



電磁相互作用の1次では、

$$S_e^{(1)} = ie \int d^4x \, \bar{\psi}(x) A_e(x) \psi(x) \text{ より}$$

$$\langle f | S_e^{(1)} | i \rangle = \langle 0 | a_s(p') \, ie \int d^4x \, \bar{\psi}_-(x) A_e(x) \psi_+(x) a_r^\dagger(p) | 0 \rangle$$

$$= ie \int d^4x \int \frac{d^4k_1 d^4k_2}{(2\pi)^{4\cdot 2}} \langle 0 | a_s(p') \left(\frac{1}{\sqrt{V}} a_\ell^\dagger(k_1) \bar{u}_\ell(k_1) e^{ik_1 x} \right) \gamma^\mu \left(\frac{1}{\sqrt{V}} a_m(k_2) u_m(k_2) e^{-ik_2 x} \right) a_r^\dagger(p) | 0 \rangle$$

$$\times \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot x} A_{e\mu}(q)$$

$$= \frac{ie}{V} \int d^4x \int \frac{d^4k_1 d^4k_2}{(2\pi)^{4\cdot 2}} \cdot (2\pi)^4 \delta_{s\ell} \delta^{(4)}(p' - k_1) \bar{u}_\ell(k_1) e^{ik_1 x}$$

$$\times \gamma^\mu \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot x} A_{e\mu}(q) \times (2\pi)^4 \delta_{mr} \delta^{(4)}(k_2 - p) u_m(k_2) e^{-ik_2 x}$$

$$= \frac{ie}{V} \int d^4x \int \frac{d^3q}{(2\pi)^3} \bar{u}_s(p') e^{ip'x} \, e^{iq\cdot x} \gamma^\mu A_{e\mu}(q) \, u_m(p) e^{-ipx}$$

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ここで、

$$A_e^\mu(x) = \left(\frac{Ze}{4\pi|x|}, 0, 0, 0 \right)$$ であり、

$$A_e^0(q) = \int d^3x \, e^{-iq\cdot x} \frac{Ze}{4\pi|x|} = \frac{Ze}{4\pi} \int d^3x \, \frac{e^{-iq\cdot x}}{|x|}$$

$$= \frac{Ze}{4\pi} \int_0^\infty dr \, r^2 \int_{-1}^1 dz \int_0^{2\pi} d\varphi \, \frac{e^{-i|q|rz}}{r}$$

$$= \frac{Ze}{2} \int_0^\infty dr \, r \left[\frac{e^{-i|q|rz}}{-i|q|r} \right]_{-1}^1 = \frac{iZe}{2|q|} \int_0^\infty dr \left(e^{-i|q|r} - e^{i|q|r} \right)$$

$$= \frac{iZe}{2|q|} \int_0^\infty dr \left(-2i \sin|q|r \right) = \frac{Ze}{|q|} \int_0^\infty dr \sin|q|r$$

ここで、積分が収束するように $\lim_{\varepsilon\to 0} e^{-\varepsilon r} = 1$ をはさむと、

$$A_e^0(q) = \lim_{\varepsilon\to 0} \frac{iZe}{2|q|} \int_0^\infty dr \left(e^{(-i|q|-\varepsilon)r} - e^{(i|q|-\varepsilon)r} \right)$$

$$= \lim_{\varepsilon\to 0} \frac{iZe}{2|q|} \left[\frac{e^{(-i|q|-\varepsilon)r}}{-i|q|-\varepsilon} - \frac{e^{(i|q|-\varepsilon)r}}{i|q|-\varepsilon} \right]_0^\infty$$

$$= \lim_{\varepsilon\to 0} \frac{iZe}{2|q|} \left(-\frac{1}{-i|q|-\varepsilon} + \frac{1}{i|q|-\varepsilon} \right)$$

$$= \frac{iZe}{2|q|} \cdot \frac{2}{i|q|} = \frac{Ze}{|q|^2}$$

$$\therefore \boxed{A_e^0(q) = \frac{Ze}{|q|^2}}$$

よって、

$$\langle f|S_e^{(1)}|i\rangle = \frac{ie}{V}\int d^4x\int\frac{d^3q}{(2\pi)^3}\cdot e^{i(p'^0-p^0)x^0}e^{i(\mathbb{p}'+\mathbb{p}+\mathbb{q})\cdot x}\bar{u}_s(p')\gamma^0 A_{e0}(\mathbb{q})u_r(p)$$

$$= \frac{ie}{V}\int d^4x\int\frac{d^3q}{(2\pi)^3}e^{i(p'^0-p^0)x^0}e^{i(-\mathbb{p}'+\mathbb{p}+\mathbb{q})\cdot x}\frac{Ze}{|\mathbb{q}|^2}\bar{u}_s(p')\gamma^0 u_r(p)$$

$$= \frac{iZe^2}{V}\int\frac{d^3q}{(2\pi)^3}\cdot(2\pi)\delta(E_{p'}-E_p)\times(2\pi)^3\delta^{(3)}(-\mathbb{p}'+\mathbb{p}+\mathbb{q})\frac{1}{|\mathbb{q}|^2}\bar{u}_s(p')\gamma^0 u_r(p)$$

$$= \frac{iZe^2}{V}\bar{u}_s(p')\gamma^0 u_r(p)\frac{1}{|\mathbb{p}'-\mathbb{p}|^2}\times(2\pi)\delta(E_{p'}-E_p).$$

$E_f=E_{p'}$, $E_i=E_p$ と書く。

全断面積を計算するには、初期状態のスピン平均と終状態のスピン和をとる必要がある。

よって、

$$M_{fi} = \frac{1}{2}\sum_r\sum_s\left(\frac{Ze^2}{V}\right)^2|\bar{u}_s(p')\gamma^0 u_r(p)|^2\cdot\frac{1}{|\mathbb{p}'-\mathbb{p}|^4}\times(2\pi)^2[\delta(E_f-E_i)]^2$$

ここで、

$$(2\pi)^2[\delta(E_f-E_i)]^2 = (2\pi)^2\delta(0)\delta(E_f-E_i)$$

$$= (2\pi)\int_{-\infty}^{\infty}dt\, e^{i\cdot 0\cdot t}\delta(E_f-E_i)$$

$$= \lim_{T\to\infty}(2\pi)\int_{-\frac{T}{2}}^{\frac{T}{2}}dt\,\delta(E_f-E_i)$$

$$= \lim_{T\to\infty}(2\pi)\cdot T\,\delta(E_f-E_i)$$

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また、

$$\frac{1}{2}\sum_{r}\sum_{s}\left|\bar{u}_s(p')\gamma^0 u_r(p)\right|^2$$

$$=\frac{1}{2}\sum_{r,s}\bar{u}_s^*(p')_a(\gamma^0)_{ab}u_r^*(p)_b\,\bar{u}_s(p')_c(\gamma^0)_{cd}u_r(p)_d$$

$$=\frac{1}{2}\sum_{r,s}u_s(p')_e(\gamma^0)_{ea}(\gamma^0)_{ab}u_r^*(p)_b\,\bar{u}_s(p')_c(\gamma^0)_{cd}u_r(p)_d$$

$$\boxed{\sum_s u_s(p)\bar{u}_s(p)=\frac{\not{p}+m}{2E_p}}$$

$$=\frac{1}{2}\sum_{r,s}u_s(p')_e\bar{u}_s(p')_c(\gamma^0)_{cd}u_r(p)_d\bar{u}_r(p)_a(\gamma^0)_{ae}$$

$$=\frac{1}{2}\cdot\frac{1}{2E_{p'}}\cdot\frac{1}{2E_p}\mathrm{Tr}\left[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0\right]$$

$$=\frac{1}{8E_{p'}E_p}\mathrm{Tr}\left[(p'^0\gamma_0+p'^j\gamma_j+m)\gamma^0(p_0-\gamma^0\gamma^i p_i+\gamma^0 m)\right]$$

$$\{\gamma^i,\gamma^j\}=2g^{ij}$$

$$=\frac{1}{8E_{p'}E_p}\mathrm{Tr}\left[p'_0p_0-\gamma^j\gamma^i p'_j p_i+m^2\right]$$

$$=\frac{1}{8E_{p'}E_p}\mathrm{Tr}\left[p'_0p_0-p'^i p_i+m^2\right]=\frac{1}{8E_{p'}E_p}\mathrm{Tr}\left[E_{p'}E_p+\mathbf{p}'\cdot\mathbf{p}+m^2\right]$$

$$=\frac{E_{p'}E_p+\mathbf{p}'\cdot\mathbf{p}+m^2}{2E_{p'}E_p}$$

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反跳 (recoil) が無い場合、運動量の大きさは保存しているので、

$E_{p'} = E_p = E, \quad |\mathbb{P}'| = |\mathbb{P}|$ が成り立つ。

$\mathbb{P}\cdot\mathbb{P}' = |\mathbb{P}|^2\cos\theta, \quad |\mathbb{P}'-\mathbb{P}|^2 = 2|\mathbb{P}|^2 - 2|\mathbb{P}|^2\cos\theta = 2|\mathbb{P}|^2(1-\cos\theta)$

$$= 4|\mathbb{P}|^2\sin^2\frac{\theta}{2}$$

よって、

$$\frac{1}{2}\sum_{\text{spin}}|\langle f|S_e^{(1)}|i\rangle|^2 = \frac{(Ze^2)^2}{V^2}\cdot\frac{E^2+|\mathbb{P}|^2\cos\theta+m^2}{2E^2}\cdot\frac{1}{16|\mathbb{P}|^4\sin^4\frac{\theta}{2}}\cdot 2\pi\cdot T\delta(E_f-E_i)$$

$$= \frac{(Ze^2)^2}{32V^2|\mathbb{P}|^4\sin^4\frac{\theta}{2}}\frac{E^2+|\mathbb{P}|^2\cos\theta+E^2-|\mathbb{P}|^2}{E^2}\times 2\pi T\delta(E_f-E_i)$$

$$= \frac{(Ze^2)^2}{32V^2|\mathbb{P}|^4\sin^4\frac{\theta}{2}}\left(2-\frac{|\mathbb{P}|^2}{E^2}(1-\cos\theta)\right)\times 2\pi T\delta(E_f-E_i)$$

$$= \frac{(Ze^2)^2}{32V^2|\mathbb{P}|^4\sin^4\frac{\theta}{2}}\left(2-2\frac{|\mathbb{P}|^2}{E^2}\sin^2\frac{\theta}{2}\right)\times 2\pi T\delta(E_f-E_i)$$

$$= \frac{(Ze^2)^2}{16V^2|\mathbb{P}|^4\sin^4\frac{\theta}{2}}\left(1-\frac{|\mathbb{P}|^2}{E^2}\sin^2\frac{\theta}{2}\right)\times 2\pi T\delta(E_f-E_i)$$

よって、全断面積は、

$$\sigma = \frac{V\cdot 2\pi}{v_{rel}}\cdot V\int\frac{d^3\mathbb{P}'}{(2\pi)^3}\frac{(Ze^2)^2}{16V^2|\mathbb{P}|^4\sin^4\frac{\theta}{2}}\left(1-\frac{|\mathbb{P}|^2}{E^2}\sin^2\frac{\theta}{2}\right)\cdot\delta(E_f-E_i)$$

$$= \int_0^\infty d|\mathbb{P}'|\,|\mathbb{P}'|^2\int d\Omega\,\frac{(Ze^2)^2}{(4\pi)^2\cdot 4|\mathbb{P}|^4\sin^4\frac{\theta}{2}\underset{v_{rel}}{\wedge}}\left(1-\frac{|\mathbb{P}'|^2}{E^2}\sin^2\frac{\theta}{2}\right)\delta(E_f-E_i)$$

よって、微分断面積は、

$E^2 = |\mathbb{P}|^2+m^2$.

$EdE = |\mathbb{P}|d|\mathbb{P}|$

$$\frac{d\sigma}{d\Omega} = \int_0^\infty d|\mathbb{P}'|\,\frac{(Z\alpha)^2}{4|\mathbb{P}'|^2\sin^4\frac{\theta}{2}\underset{v_{rel}}{\wedge}}\left(1-\frac{|\mathbb{P}'|^2}{E^2}\sin^2\frac{\theta}{2}\right)\delta(E_f-E_i)$$

$$= \int_m^\infty dE_f\,\frac{(Z\alpha)^2}{4v_{rel}|\mathbb{P}|^2\sin^4\frac{\theta}{2}}\left[1-\frac{|\mathbb{P}|^2}{E^2}\sin^2\frac{\theta}{2}\right]\cdot\frac{E}{|\mathbb{P}|}\delta(E_f-E_i)$$

$|\mathbb{P}| = Ev.$

$$= \frac{(Z\alpha)^2}{4E^2v_{rel}^4\sin^4\frac{\theta}{2}}\left(1-v_{rel}^2\sin^2\frac{\theta}{2}\right)$$