$\boxed{\text{WKB 近似}}$

$\psi = C\exp\left(i\frac{S}{\hbar}\right)$ と書く。 また、$S = S_0 + \hbar S_1 + \cdots$ と展開し、$\hbar$の1次まで採る。

シュレーディンガー方程式をたてると、

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\frac{d\psi}{dx} = C\frac{iS'}{\hbar}\exp\left(i\frac{S}{\hbar}\right) = \frac{iS'}{\hbar}\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{iS''}{\hbar}\psi + \left(\frac{iS'}{\hbar}\right)^2\psi = \left[\frac{i}{\hbar}S'' - \frac{(S')^2}{\hbar^2}\right]\psi$$

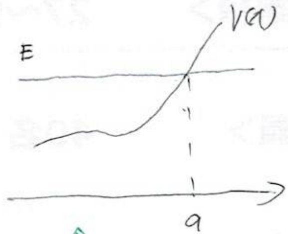
$$\Leftrightarrow\ -\frac{\hbar^2}{2m}\left[\frac{i}{\hbar}S'' - \frac{(S')^2}{\hbar^2}\right]\psi = (E-V(x))\psi$$

代入

$$\Leftrightarrow\ -i\hbar S'' + (S')^2 = 2m(E-V(x))$$

$$\Leftrightarrow\ -i\hbar(S_0'' + \hbar S_1'') + (S_0' + \hbar S_1')^2 = 2m(E-V(x))$$

$$\Leftrightarrow\ \underbrace{(S_0')^2 - 2m(E-V(x))}_{0} + \hbar(\underbrace{2S_0'S_1' - iS_0''}_{0}) + \cdots = 0.$$



このイメージで

$$\therefore\ \boxed{\begin{cases}(S_0')^2 = 2m(E-V(x)) \\ 2S_0'S_1' - iS_0'' = 0\end{cases}}$$

E>Vのとき、 $S_0' = \pm\sqrt{2m(E-V(x))} = \pm\hbar k(x) \longrightarrow S_0 = \pm\hbar\int_x^a dx' k(x')$

$S_1' = \frac{i}{2}\cdot\frac{k'(x)}{k(x)} \longrightarrow S_1 = C_1 \cdot i\log\sqrt{k(x)} =$

よって、 $\psi = C\cdot\exp\left[\pm i\int_a^x dx' k(x') - \frac{C_1}{\hbar}\log\sqrt{k(x)}\right]$

$$= \frac{C}{\sqrt{k(x)}}\exp\left[\pm i\int_a^x dx'\frac{\sqrt{2m(E-V(x))}}{\hbar}\right]$$

一方で、$E < V(x)$ のとき、

$$(S_0')^2 = -2m(V(x)-E) \quad \Longleftrightarrow \quad S_0' = \pm i\sqrt{2m(V(x)-E)} = \pm i\hbar\kappa(x)$$

$$S_1' = \frac{i}{2}\cdot\frac{\kappa'(x)}{\kappa(x)} \quad \text{より} \quad S_1 = i\,C_1 \log\sqrt{\kappa(x)}$$

$$\Downarrow$$

$$\psi = \frac{D}{\sqrt{\kappa(x)}} \exp\left[\pm\int_a^x dx'\,\kappa(x')\right]$$

$$\boxed{\eta(a,x) = \int_a^x dx' \frac{\sqrt{2m|E-V(x)|}}{\hbar}}$$ を用いると、

$$\boxed{\begin{aligned} &E > V \text{ のとき} \quad \psi = \frac{C_+}{\sqrt{\kappa(x)}}\exp\left[i\eta(a,x)\right] + \frac{C_-}{\sqrt{\kappa(x)}}\exp\left[-i\eta(a,x)\right] \\ &E < V \text{ のとき} \quad \psi = \frac{D_+}{\sqrt{\kappa(x)}}\exp\left[\eta(a,x)\right] + \frac{D_-}{\sqrt{\kappa(x)}}\exp\left[-\eta(a,x)\right] \end{aligned}}$$