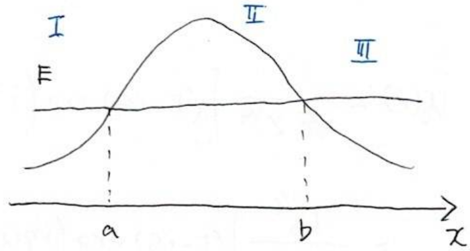
トンネル効果

図のようなポテンシャル障壁に対して
$x<a$ の領域から x軸正の向きに
粒子が入射し、$x>b$ の領域に
透過する確率を求める。



領域I, II では、WKB近似解は

$$u_{I}(x) = \frac{C_+}{\sqrt{k(x)}} \exp\left(i\eta(a,x)\right) + \frac{C_-}{\sqrt{k(x)}} \exp\left(-i\eta(a,x)\right)$$

$$u_{II}(x) = \frac{D_+}{\sqrt{k(x)}} \exp\left(\eta(a,x)\right) + \frac{D_-}{\sqrt{k(x)}} \exp\left(-\eta(a,x)\right)$$

係数の関係 $C_\pm = \left(D_- \pm \frac{i}{2} D_+\right) \exp\left(\mp i\frac{\pi}{4}\right)$

領域II, III について、WKB近似解は

$$u_{II}(x) = \frac{D_{2+}}{\sqrt{k(x)}} \exp\left(\eta(x,b)\right) + \frac{D_{2-}}{\sqrt{k(x)}} \exp\left(-\eta(x,b)\right)$$

$$u_{III}(x) = \frac{C_{2+}}{\sqrt{k(x)}} \exp\left(-i\eta(b,x)\right) + \frac{C_{2-}}{\sqrt{k(x)}} \exp\left(i\eta(b,x)\right)$$

係数の関係 $C_{2\pm} = \left(D_{2-} \mp \frac{i}{2} D_{2+}\right) \exp\left(\pm i\frac{\pi}{4}\right)$

進行波のみ考えるので、

$C_{2+} = 0$ とする $\Rightarrow$ $D_{2-} = \frac{i}{2} D_{2+}$

C_{2-} に代入

$$C_{2-} = C = 2D_{2-} \exp\left(-i\frac{\pi}{4}\right)$$

$$\therefore \boxed{D_{2-} = \frac{C}{2} \exp\left(+i\frac{\pi}{4}\right)}$$

また、

$$D_{2+} = -2i D_{2-} = -i C \exp\left(+i\frac{\pi}{4}\right)$$

$-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$ より

$= \exp\left(-i\frac{\pi}{2}\right)$

$$\therefore \boxed{D_{2+} = C\exp\left(-i\frac{\pi}{4}\right)}$$

これにより、WKB近似解は

$$\boxed{\begin{aligned} u_{II}(x) &= \frac{C}{\sqrt{k(x)}}\left[\exp\left(-\eta(b,x) - i\frac{\pi}{4}\right) + \frac{1}{2}\exp\left(\eta(b,x) + i\frac{\pi}{4}\right)\right] \\ u_{III}(x) &= \frac{C}{\sqrt{k(x)}}\exp\left(i\eta(b,x)\right) \end{aligned}}$$

係数の関係　$D_{2+} = C\exp\left(-i\frac{\pi}{4}\right)$, $D_{2-} = \frac{C}{2}\exp\left(i\frac{\pi}{4}\right)$

領域IIの波動関数の係数を比べて、

$$C\exp\left(-\eta(b,x) - i\frac{\pi}{4}\right) = D_-\exp\left(-\eta(a,x)\right)$$

$$\begin{aligned} D_- &= C\exp\left(\eta(a,x) - \eta(b,x) - i\frac{\pi}{4}\right) \\ &= C\exp\left(\eta(a,b) - i\frac{\pi}{4}\right) \\ &= \underline{C\exp\left(P - i\frac{\pi}{4}\right)} \end{aligned}$$

$P = \int_a^b dx\, k(x) =$

$$\frac{1}{2}C\exp\left(\eta(b,x) + i\frac{\pi}{4}\right) = D_+\exp\left(\eta(a,x)\right)$$

$$\begin{aligned} D_+ &= \frac{C}{2}\exp\left(-\eta(a,x) + \eta(b,x) + i\frac{\pi}{4}\right) \\ &= \underline{\frac{C}{2}\exp\left(-P + i\frac{\pi}{4}\right)} \end{aligned}$$

関係

$$C_\pm = \left(D_- \pm \frac{i}{2} D_+\right) \exp\left(\mp i \frac{\pi}{4}\right) \text{ より}$$

$$D_+ = \frac{C}{2} \exp\left(-P + i\frac{\pi}{4}\right)$$

$$D_- = C \exp\left(P - i\frac{\pi}{4}\right)$$

$$C_+ = \left[\frac{C}{\cancel{2}} \exp\left(+P - i\frac{\pi}{4}\right) + \frac{iC}{4} \exp\left(-P + i\frac{\pi}{4}\right)\right] \exp\left(-i\frac{\pi}{4}\right)$$

$$= C\left[\exp\left(P - \frac{\pi}{2}\right) + \frac{i}{4} \exp(-P)\right]$$

$$= -iC\left[e^P - \frac{1}{4} e^{-P}\right]$$

$$C_- = \left[C \exp\left(P - i\frac{\pi}{4}\right) - \frac{iC}{4} \exp\left(-P + i\frac{\pi}{4}\right)\right] \exp\left(i\frac{\pi}{4}\right)$$

$$= C\left[e^P - \frac{i}{4} \exp\left(-P + \underbrace{i\frac{\pi}{2}}_{e^{+i\frac{\pi}{2}} = +i}\right)\right] = C\left[e^P + \frac{1}{4} e^{-P}\right]$$

入射波の wave function

$$u_{in}(x) = \frac{C_+}{\sqrt{k(x)}} \exp\left(i\eta(a, x)\right)$$

$$u'_{in}(x) = -\frac{C_+}{2} \cdot \frac{k'(x)}{(k(x))^{3/2}} \exp\left(i\eta(a, x)\right) + \frac{C_+}{\sqrt{k(x)}} \cdot ik(x) \exp\left(i\eta(a, x)\right)$$

$$= -\frac{C_+ k'(x)}{2\sqrt{k(x)^3}} \exp\left(i\eta(a, x)\right) + iC_+\sqrt{k(x)} \exp\left(i\eta(a, x)\right)$$

$$j_i = -\frac{i\hbar}{2m}\left(\psi^*\psi' - \psi^{*\prime}\psi\right)$$

よって

$$j_{in} = -\frac{i\hbar}{2m} \cdot \left[-\frac{|C_+|^2 \cancel{k'(x)}}{\cancel{2}(k(x))^2} + i|C_+|^2 + \frac{|C_+|^2 \cancel{k'(x)}}{\cancel{2}(k(x))^2} + i|C_+|^2\right]$$

$$= \frac{\hbar}{m}|C_+|^2 = \frac{\hbar}{m}|C|^2 \cdot \left(e^P - \frac{1}{4} e^{-P}\right)^2$$

透過波は

$$u_{trans}(x) = \frac{C}{\sqrt{k(x)}} \exp\left(i\eta(b,x)\right) \text{ より}$$

$$u'_{trans}(x) = \frac{Ck'(x)}{2\sqrt{k(x)^3}} \exp\left(i\eta(b,x)\right) + iC\sqrt{k(x)} \exp\left(i\eta(b,x)\right)$$

よって、

$$j_{trans} = -\frac{i\hbar}{2m}\left[\frac{|C|^2 k'(x)}{\cancel{2(k(x))^2}} + i|C|^2 - \frac{|C|^2 k'(x)}{\cancel{2(k(x))^2}} + i|C|^2\right]$$

$$= +\frac{\hbar}{m}|C|^2$$

これらにより、透過率は

$$T = \frac{j_{trans}}{j_{in}} = \frac{\frac{\hbar}{m}|C|^2}{\frac{\hbar}{m}|C|^2\left(e^{P} - \frac{1}{4}e^{-P}\right)^2}$$

$$= \frac{e^{-2P}}{\left(1 - \frac{1}{4}e^{-2P}\right)^2} \simeq e^{-2P}$$

← WKB近似は
$P = \eta(a,b) \gg 1$ で
良い近似となる。

$$\therefore \boxed{T = e^{-2P} = \exp\left[-\frac{2}{\hbar}\int_a^b dx\sqrt{2m(V(x)-E)}\right]}$$

ガモフの透過因子