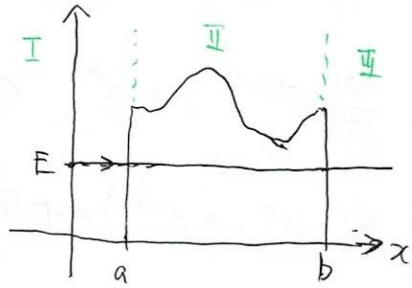
$\boxed{\alpha\text{崩壊}}$

少し大ざっぱな評価をする.

$$V(x) = \begin{cases} 0 & (x \leqq a) \\ V(x) & (a \leqq x \leqq b) \\ 0 & (b \leqq x) \end{cases}$$ を考えると、



$$\psi_{\mathrm{I}} = A e^{ikx} + B e^{-ikx} \qquad (x \leqq a)$$

$$\psi_{\mathrm{II}} = \frac{C}{\sqrt{\kappa(x)}} e^{\int_a^x dx' \kappa(x')} + \frac{D}{\sqrt{\kappa(x)}} e^{-\int_a^x dx' \kappa(x')} \qquad (a < x < b)$$

$$\psi_{\mathrm{II}} = F e^{ikx}$$

接続条件

$$A e^{ika} + B e^{-ika} = \frac{C+D}{\sqrt{\kappa(a)}} \quad \cdots ①$$

$$\frac{C}{\sqrt{\kappa(b)}} e^{\eta(a,b)} + \frac{D}{\sqrt{\kappa(b)}} e^{-\eta(a,b)} = F e^{ikb} \quad \cdots ②$$

$$\psi_{\mathrm{I}}' = ik\left(A e^{ikx} - B e^{-ikx}\right)$$

$$\psi_{\mathrm{II}}' = -\frac{1}{2} C \kappa(x)^{-\frac{3}{2}} e^{\eta(a,x)} + C\sqrt{\kappa(x)}\, e^{\eta(a,x)}$$
$$- \frac{1}{2} D \kappa(x)^{-3/2} e^{-\eta(a,x)} + D\left(-\sqrt{\kappa(x)}\right) e^{-\eta(a,x)}$$
$$= C\sqrt{\kappa(x)}\left(1 - \frac{1}{2\kappa(x)^2}\right) e^{\eta(a,x)} - D\sqrt{\kappa(x)}\left(1 + \frac{1}{2\kappa(x)^2}\right) e^{-\eta(a,x)}$$

$$\psi_{\mathrm{III}}' = ikF e^{ikx}$$

$$ik\left(A e^{ika} - B e^{-ika}\right) = C\sqrt{\kappa(a)}\left(1 - \frac{1}{2\kappa(a)^2}\right) - D\sqrt{\kappa(a)}\left(1 + \frac{1}{2\kappa(a)^2}\right)$$
$$= (C-D)\sqrt{\kappa(a)} - \frac{C+D}{2(\kappa(a))^{3/2}}$$

ポテンシャル中では x が大きくなるにつれて減衰するため、~~[crossed out]~~ としてしまって $C=0$

~~[crossed out]~~

$$Ae^{ika}+Be^{-ika}=\frac{D}{\sqrt{\kappa(a)}} \quad ①$$

$$\frac{D}{\sqrt{\kappa(a)}}e^{-\eta(a,b)}=Fe^{ikb} \quad ②$$

$$ik\left(Ae^{ika}-Be^{-ika}\right)=-D\sqrt{\kappa(a)}\left(1+\frac{1}{2\kappa(a)^2}\right) \quad ③$$

$$-D\sqrt{\kappa(b)}\left(1+\frac{1}{2\kappa(b)^2}\right)e^{-\eta(a,b)}=ikFe^{ikb} \quad ④$$

$$\underline{Ae^{ika}+Be^{-ika}=Fe^{ikb+\eta(a,b)}} \quad (\text{From } ①+②)$$

$B=0$

③, ④ より

$$ik\left(Ae^{ika}-Be^{-ika}\right)=ikFe^{ikb+\eta(a,b)} \Leftrightarrow \underline{Ae^{ika}-Be^{-ika}=Fe^{ikb+\eta(a,b)}}$$

$$\Rightarrow \quad Ae^{ika}=Fe^{ikb+\eta(a,b)}$$

$$\Leftrightarrow \quad \frac{F}{A}=\exp\left(ik(a-b)-\int_a^b dx'\kappa(x')\right)$$

$$\boxed{T\simeq\frac{|F|^2}{|A|^2}=\exp\left(-2\int_a^b dx'\kappa(x')\right)}$$

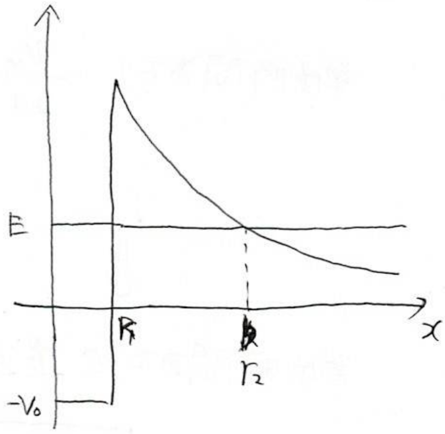
透過率

$$T=\exp\left(-\frac{2}{\hbar}\int_a^b dx'\sqrt{2m(V(x)-E)}\right)$$

クーロンポテンシャルの場合.

$$P = \int_a^b dx' \frac{\sqrt{2m(V_c(r')-E)}}{\hbar}$$

$$= \int_a^b d\cancel{x'} \frac{\sqrt{2mE(b/r-1)}}{\hbar}$$



$r=r_2$ において

$$V_c(b) = \frac{Z_\alpha Z_D \hbar c\alpha}{b} = E$$

$$V_c(r) = \frac{Z_\alpha Z_D \hbar c\alpha}{r} = \underbrace{\frac{Z_\alpha Z_D \hbar c\alpha}{b}}_{E} \cdot \frac{b}{r} = \frac{Eb}{r}$$

$$= \frac{\sqrt{2mE}}{\hbar}\int_a^b dr \sqrt{\frac{b}{r}-1}$$

$r = b\cos^2\theta$ とおくと. $0 \leqq \theta \leqq \theta_0$ ただし. $\underline{a = b\cos^2\theta_0}$

$r = \frac{b}{2}(1+\cos 2\theta)$ より $dr = -\frac{b}{2}\cdot\sin 2\theta \cdot 2\, d\theta$

$$= -b\sin 2\theta\, d\theta$$

$$= -2b\sin\theta\cos\theta\, d\theta$$

$$= \frac{\sqrt{2mE}}{\hbar}\int_{\theta_0}^0 (-2b\sin\theta\cos\theta\, d\theta)\cdot\sqrt{\frac{b}{b\cos^2\theta}-1}$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin^2\theta$

$$= \frac{\sqrt{2mE}}{\hbar}\int_0^{\theta_0} d\theta\cdot 2b\sin\theta\cos\theta\sqrt{\frac{1-\cos^2\theta}{\cos^2\theta}}$$

$a = b\cos^2\theta_0$ より

$\cos^2\theta_0 = \frac{a}{b} = \varepsilon^2.$

$$= \frac{\sqrt{2mE}}{\hbar}\int_0^{\theta_0} d\theta\, 2b\sin^2\theta = \frac{\sqrt{2mE}}{\hbar}b\int_0^{\theta_0} d\theta\,(1-\cos 2\theta)$$

$\underline{\underline{\theta_0 = \cos^{-1}\varepsilon.}}$

$\sin\theta\cos\theta = \varepsilon\sqrt{1-\varepsilon^2}$

$$= \frac{\sqrt{2mE}}{\hbar}b\cdot\left[\theta_0 - \frac{1}{2}\sin 2\theta_0\right]$$

$$= \frac{\sqrt{2mE}}{\hbar}b\left(\cos^{-1}\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right)$$

原子核内でのα粒子の速さをv_{in}とすると、

単位時間あたり $\frac{v_{in}}{2a}$ 回衝突する

ポテンシャル障壁に

$\Downarrow$

単位時間あたりの透過確率は

$$\boxed{w = \frac{v_{in}}{2a} e^{-2P}}$$

微小時間 dt の間に崩壊する粒子数: $N(t)w\,dt$ より

$t+dt$ での粒子数は

$$N(t+dt) = N(t) - N(t)w\,dt$$

$$\Leftrightarrow \boxed{\frac{dN(t)}{dt} = -wN(t)} \quad \rightarrow \quad \boxed{N(t) = N(0)e^{-wt}}$$

個数 $N(t)$ が半分になるのに要する時間 = 半減期

$T_{1/2}$ を求める.

$$\frac{N(t+T_{1/2})}{N(t)} = e^{-wT_{1/2}} = \frac{1}{2}, \quad \Leftrightarrow e^{wT_{1/2}} = 2 \qquad \log$$

$$\Leftrightarrow wT_{1/2} = \log 2$$

$$\therefore T_{1/2} = \frac{\log 2}{w}$$

$$= \frac{2a\log 2}{v_{in}} e^{2P}$$

v_{in} を十分遠方の速さとすると、

$$\frac{m v_{in}^2}{2} \simeq E \quad \rightarrow \quad v_{in} = \sqrt{\frac{2E}{m}}$$

クーロンポテンシャルの場合

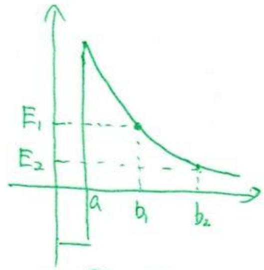
$$P = \frac{\sqrt{2mE}}{\hbar} b\left(\cos^{-1}\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right)$$

$$\varepsilon = \cos\theta .$$

よって

$$T_{1/2} = \sqrt{\frac{m}{2E}} \cdot 2a\log 2 \; \exp\left[\frac{2\sqrt{2mE}}{\hbar} b\left(\cos^{-1}\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right)\right]$$

$$= \sqrt{\frac{2m}{E}}\, a\log 2 \; \exp\left[\frac{2\sqrt{2mE}}{\hbar} b\left(\cos^{-1}\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right)\right]$$



アルファ粒子のエネルギーは ポテンシャル障壁よりも十分小さいと考えられる。→ $\boxed{\varepsilon \ll 1}$

$$\boxed{\cos^{-1}\varepsilon = \frac{\pi}{2} - \varepsilon - \frac{\varepsilon^3}{6} - \frac{3}{40}\varepsilon^5 - \frac{5}{112}\varepsilon^7 - \cdots}$$

for $-1 < \varepsilon < 1$

また、$\varepsilon\sqrt{1-\varepsilon^2} \simeq \varepsilon\left(1 - \frac{\varepsilon^2}{2}\right)$

$$= \boxed{\varepsilon - \frac{\varepsilon^3}{2}} + \cdots$$

$O(\varepsilon^3)$ までとると、

$$T_{1/2} = \sqrt{\frac{2m}{E}}\, a\log 2 \; \exp\left[\frac{2\sqrt{2mE}}{\hbar} b\left\{\frac{\pi}{2} - \varepsilon - \frac{\varepsilon^3}{6} - \varepsilon + \frac{\varepsilon^3}{2}\right\}\right]$$

$$= \sqrt{\frac{2m}{E}}\, a\log 2 \; \exp\left[\frac{2\sqrt{2mE}}{\hbar} b\left(\frac{\pi}{2} - 2\varepsilon + \frac{1}{3}\varepsilon^3\right)\right]$$

$$E = \frac{Z_\alpha Z_D \hbar c \alpha}{b}$$

$$\Leftrightarrow b = \frac{Z_\alpha Z_D \hbar c \alpha}{E}$$

$$\frac{2\sqrt{2mE}}{\hbar} \cdot \frac{Z_\alpha Z_D \hbar c \alpha}{E}$$

$$= \frac{2\sqrt{2mc^2}\, Z_\alpha Z_D \alpha}{\sqrt{E}}$$

$$= \sqrt{\frac{2m}{E}}\, a \log 2 \ \exp\left[\frac{\pi\sqrt{2mc^2}\, Z_\alpha Z_D \alpha}{\sqrt{E}} - 4\sqrt{\frac{2mc^2 Z_\alpha Z_D \alpha a}{\hbar c}} + \frac{2}{3}\sqrt{\frac{2mc^2 a^3}{Z_\alpha Z_D \alpha (\hbar c)^3}}\, E \right]$$

$$= \sqrt{\frac{2m}{E}}\, a \log 2 \cdot \exp\left[\frac{\pi\sqrt{2mc^2}\, \gamma}{\sqrt{E}} - 4\sqrt{\frac{2mc^2 \gamma a}{\hbar c}} + \frac{2}{3}\sqrt{\frac{2mc^2 a^3}{\gamma (\hbar c)^3}}\, E \right]$$

$\gamma = Z_\alpha Z_D \alpha$

I used.

$$\varepsilon^2 = \frac{a}{b} = \frac{aE}{Z_\alpha Z_D \hbar c \alpha}$$

$$\rightarrow \varepsilon = \left(\frac{aE}{Z_\alpha Z_D \hbar c \alpha} \right)^{1/2}$$