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フェルミオン－フェルミオン散乱（Møller散乱）

単位時間あたりの遷移確率は

$$w_{fi} = \lim_{\substack{t_1 \to -\infty \\ t_2 \to \infty}} \frac{|\langle f|U(t_2,t_1)-1|i\rangle|^2}{t_2 - t_1} = \frac{|\langle f|S-1|i\rangle|^2}{t_2 - t_1}$$

相互作用の1次の項を考えると

$$\langle f|U(\infty,-\infty)-1|i\rangle = -i\int_{-\infty}^{\infty} dt\, e^{i(E_f-E_i)t}\langle f|H_{int}|i\rangle$$

$$= -2\pi i\,\delta(E_f-E_i)\langle f|H_{int}|i\rangle$$

相互作用の2次以上の項も含めた、エネルギー・運動量を保存する作用を集めると、

$$\langle f|S-1|i\rangle = -2\pi i \underbrace{\delta(E_f-E_i)}_{\text{エネルギー保存}} \times (2\pi)^3 \underbrace{\delta^{(3)}(\mathbb{P}_f-\mathbb{P}_i)}_{\text{運動量保存}} \langle f|\mathcal{T}|i\rangle$$

$$|\langle f|S-1|i\rangle|^2 = (2\pi)^8 \{\delta(E_f-E_i)\}^2 \{\delta^{(3)}(\mathbb{P}_f-\mathbb{P}_i)\}^2 |\langle f|\mathcal{T}|i\rangle|^2$$

$$= (2\pi)^4 \underbrace{\int_{-\infty}^{\infty} dt\, e^{i\cdot 0\cdot t}}_{\lim_{t\to\infty} t} \delta(E_f-E_i) \underbrace{\int d^3x\, e^{i 0\cdot x}}_{\lim_{V\to\infty} V} \delta^{(3)}(\mathbb{P}_f-\mathbb{P}_i) |\langle f|\mathcal{T}|i\rangle|^2$$

$$= (2\pi)^4 \lim_{\substack{t\to\infty \\ V\to\infty}} tV\, \delta(E_f-E_i)\,\delta^{(3)}(\mathbb{P}_f-\mathbb{P}_i)\,|\langle f|\mathcal{T}|i\rangle|^2$$

2粒子 → n粒子の過程

$$\sigma = \overbrace{\frac{V}{(2\pi)^3}\int d^3q_1 \cdots \frac{V}{(2\pi)^3}\int d^3q_n}^{n\text{個}} \frac{V^2 (2\pi)^4}{v_{rel}} \sum_{spins} |\langle f|\mathcal{T}|i\rangle|^2 \delta(E_f-E_i)\,\delta^{(3)}(\mathbb{P}_f-\mathbb{P}_i)$$

$$= \frac{V^{n+2}}{(2\pi)^{3n-4} v_{rel}} \int d^3q_1 \cdots \int d^3q_n \sum_{spin} |\langle f|\mathcal{T}|i\rangle|^2 \delta(E_f-E_i)\,\delta^{(3)}(\mathbb{P}_f-\mathbb{P}_i)$$

$$= \frac{(2\pi)^{4-3n}}{\underbrace{4 v_{rel} E_{p_1} E_{p_2}}_{\text{Lorentz不変}}} \int \underbrace{\frac{d^3q_1}{2E_{q_1}}}_{} \cdots \int \underbrace{\frac{d^3q_n}{2E_{q_n}}}_{} \delta(E_f-E_i)\,\delta^{(3)}(\mathbb{P}_f-\mathbb{P}_i)$$

それぞれLorentz不変

$$\times \underbrace{V^{n+2} \cdot 2E_{p_1} \cdot 2E_{p_2} \cdot 2E_{q_1} \cdots 2E_{q_n} |\langle f|\mathcal{T}|i\rangle|^2}_{\text{全体でLorentz不変}}$$

重心系および実験室系での相対速度

相対速度 → $v_{rel} = \left| \frac{\mathbb{P}_1}{E_1} - \frac{\mathbb{P}_2}{E_2} \right|$ ← 両辺に E_1E_2 かける。

$$E_1E_2 v_{rel} = |E_2\mathbb{P}_1 - E_1\mathbb{P}_2| = \sqrt{(E_2\mathbb{P}_1 - E_1\mathbb{P}_2)^2}$$

← 粒子が散乱するとき
どちらかの静止系 or 重心系から見ると
2粒子は同一直線上で運動するため
外積が0となる。

$$= \sqrt{(E_2\mathbb{P}_1 - E_1\mathbb{P}_2)^2 - (\mathbb{P}_1 \times \mathbb{P}_2)^2}$$

$$= \sqrt{E_2^2\mathbb{P}_1^2 - 2E_1E_2\mathbb{P}_1\cdot\mathbb{P}_2 + E_1^2\mathbb{P}_2^2 - \mathbb{P}_1^2\mathbb{P}_2^2 + (\mathbb{P}_1\cdot\mathbb{P}_2)^2}$$

外積の2乗

$$(a\times b)^2 = \varepsilon_{ijk}a_jb_k\,\varepsilon_{ilm}a_lb_m$$

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})a_jb_ka_lb_m$$

$$= a^2b^2 - (a\cdot b)^2$$

$$= \sqrt{E_2^2(E_1^2 - M_1^2) - 2E_1E_2(\mathbb{P}_1\cdot\mathbb{P}_2) + E_1^2(E_2^2 - M_2^2) - (E_1^2 - M_1^2)(E_2^2 - M_2^2) + (\mathbb{P}_1\cdot\mathbb{P}_2)^2}$$

$$= \sqrt{E_1^2E_2^2 - E_2^2M_1^2 - 2E_1E_2(\mathbb{P}_1\cdot\mathbb{P}_2) + E_1^2E_2^2 - E_1^2M_2^2 - E_1^2E_2^2 + E_1^2M_2^2 + E_2^2M_1^2 - M_1^2M_2^2 + (\mathbb{P}_1\cdot\mathbb{P}_2)^2}$$

$$= \sqrt{E_1^2E_2^2 - 2E_1E_2(\mathbb{P}_1\cdot\mathbb{P}_2) + (\mathbb{P}_1\cdot\mathbb{P}_2)^2 - M_1^2M_2^2}$$

$$= \sqrt{(E_1E_2 - \mathbb{P}_1\cdot\mathbb{P}_2)^2 - M_1^2M_2^2}$$

$$= \underbrace{\sqrt{(P_1\cdot P_2)^2 - (M_1M_2)^2}}_{\text{Lorentz不変}} = B$$

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・2粒子 → 2粒子 を考えると、

$$\sigma = \frac{1}{4\cdot(2\pi)^2 v_{rel} E_{p_1} E_{p_2}} \int \frac{d^3 q_1}{2E_{q_1}} \int \frac{d^3 q_2}{2E_{q_2}} \delta(E_f - E_i)\, \delta^{(3)}(\mathbb{P}_f - \mathbb{P}_i)$$

$$\times V^4 \cdot 2E_{p_1} \cdot 2E_{p_2} \cdot 2E_{q_1} \cdot 2E_{q_2} \left| \langle f | T | i \rangle \right|^2$$

ここで、

$$\frac{1}{2E} \int_0^\infty dq_0 \frac{1}{2E} \delta(q_0 - E) = \int_0^\infty dq_0 \frac{1}{2\sqrt{q^2+m^2}} \left\{ \delta\left(q_0 - \sqrt{q^2+m^2}\right) + \underbrace{\delta\left(q_0 + \sqrt{q^2+m^2}\right)}_{0} \right\}$$

$$= \int_{-\infty}^\infty dq_0 \frac{1}{2\sqrt{q^2+m^2}} \left\{ \delta\left(q_0 - \sqrt{q^2+m^2}\right) + \delta\left(q_0 + \sqrt{q^2+m^2}\right) \right\} \theta(q_0)$$

$$= \int_{-\infty}^\infty dq_0\, \delta\left(q_0^2 - q^2 - m^2\right) \theta(q_0) = \int_{-\infty}^\infty dq_0\, \delta\left(q^2 - m^2\right) \theta(q_0)$$

デルタ関数の性質

$$\delta(x^2 - a^2) = \frac{1}{2a} \left\{ \delta(x+a) + \delta(x-a) \right\}$$

$$I = \int \frac{d^3 q_1}{2E_{q_1}} \int \frac{d^3 q_2}{2E_{q_2}} \delta(E_f - E_i)\, \delta^{(3)}(\mathbb{P}_f - \mathbb{P}_i)$$

$$= \int d^4 q_1 \int d^4 q_2\, \delta(q_1^2 - M_1^2) \underbrace{\theta(q_{10})}_{q_{10}>0} \delta(q_2^2 - M_2^2) \theta(q_{20}) \underbrace{\delta(E_f - E_i)\, \delta^{(3)}(\mathbb{P}_f - \mathbb{P}_i)}_{\delta^{(4)}(p_1 + p_2 - q_1 - q_2)}$$

$$= \int d^4 q_1\, \delta(q_1^2 - M_1^2) \underbrace{\theta(q_{10})}_{q_{10}>0} \delta\left([-q_1 + p_1 + p_2]^2 - M_2^2 \right) \underbrace{\theta(-q_{10} + E_1 + E_2)}_{q_{10} < E_1 + E_2}$$

$$= \int_0^{E_1+E_2} dq_{10} \int_0^\infty d|q_1|\, |q_1|^2 \int d\Omega\, \delta\left(-|q_1|^2 + q_{10}^2 - M_1^2\right) \delta\left([-q_1 + p_1 + p_2]^2 - M_2^2 \right)$$

$|q_1|$で積分

$$= \int_0^{E_1+E_2} dq_{10} \int_0^\infty d|q_1|\, |q_1|^2 \int d\Omega \frac{1}{2|q_1|} \left\{ \underbrace{\delta\left(\sqrt{q_{10}^2 - M_1^2} + |q_1| \right)}_{0} + \delta\left(\sqrt{q_{10}^2 - M_1^2} - |q_1| \right) \right\} \delta\left[(p_1 + p_2 - q_1)^2 - M_2^2 \right]$$

$$= \int_0^{E_1+E_2} dq_{10} \int d\Omega \frac{\sqrt{q_{10}^2 - M_1^2}}{2} \delta\left[(p_1 + p_2)^2 - 2(p_1 + p_2)\cdot q_1 + M_1^2 - M_2^2 \right]$$

ここで 重心系をとると、$\mathbb{P}_1+\mathbb{P}_2=0$ より

$$(P_1+P_2)^2=(P_{10}+P_{20})^2-(\underbrace{\mathbb{P}_1+\mathbb{P}_2}_{0})^2=(E_1+E_2)^2=S=E_{tot}^2$$

このとき、

$$I=\int_0^{E_1+E_2}d\varepsilon_{10}\int d\Omega\,\frac{\sqrt{\varepsilon_{10}^2-M_1^2}}{2}\,\delta\left[(P_1+P_2)^2-2(P_1+P_2)\cdot\varepsilon_1+M_1^2-M_2^2\right]$$

$$=\int_0^{E_1+E_2}d\varepsilon_{10}\int d\Omega\,\frac{\sqrt{\varepsilon_{10}^2-M_1^2}}{2}\,\delta\left[S-2(P_{10}+P_{20})\varepsilon_{10}+M_1^2-M_2^2\right]$$

$$=\int_0^{E_1+E_2}d\varepsilon_{10}\int d\Omega\,\frac{\sqrt{\varepsilon_{10}^2-M_1^2}}{2}\,\delta\left(S-2\sqrt{S}\,\varepsilon_{10}+M_1^2-M_2^2\right)$$

$$S-2\sqrt{S}\,\varepsilon_{10}+M_1^2-M_2^2=0$$

$$\Downarrow$$

$$\varepsilon_{10}=\frac{S+M_1^2-M_2^2}{2\sqrt{S}}$$

$$=\int_0^{\sqrt{S}}d\varepsilon_{10}\int d\Omega\,\frac{\sqrt{\varepsilon_{10}^2-M_1^2}}{2}\,\delta\left(S-2\sqrt{S}\,\varepsilon_{10}+M_1^2-M_2^2\right)$$

$$=\int d\Omega\,\frac{1}{2}\sqrt{\frac{(S+M_1^2-M_2^2)^2}{4S}-M_1^2}\cdot\frac{1}{2\sqrt{S}}$$

$$=\int d\Omega\,\frac{1}{8S}\sqrt{S^2+M_1^4+M_2^4+2SM_1^2-2SM_2^2-2M_1^2M_2^2-4SM_1^2}$$

$$=\int d\Omega\,\frac{1}{8S}\sqrt{S^2+M_1^4+M_2^4-2SM_1^2-2SM_2^2-2M_1^2M_2^2}$$

$$=\int d\Omega\,\frac{1}{8S}\sqrt{(S-M_1^2-M_2^2)^2-4M_1^2M_2^2}$$

$$=\int d\Omega\,\frac{\sqrt{(S-M_1^2-M_2^2+2M_1M_2)(S-M_1^2-M_2^2-2M_1M_2)}}{8S}$$

$$=\int d\Omega\,\frac{\sqrt{[S-(M_1-M_2)^2][S-(M_1+M_2)^2]}}{8S}$$

$$=\int d\Omega\,\frac{|\mathbb{P}_f|}{4\sqrt{S}}$$

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$v_{rel}E_{P_1}E_{P_2}$ の書き換え

$$v_{rel}E_1E_2 = B = \sqrt{(P_1\cdot P_2)^2 - m_1^2m_2^2}$$

$E_1^2 = \mathbb{P}_1^2 + m_1^2$, $E_2^2 = \mathbb{P}_2^2 + m_2^2$. $\mathbb{P}_2 = -\mathbb{P}_1$ を用いて.

$$(E_1+E_2)^2 = S = E_1^2 + 2E_1E_2 + E_2^2$$
$$= 2E_1E_2 + \mathbb{P}_1^2 + m_1^2 + \mathbb{P}_2^2 + m_2^2$$
$$\therefore\ 2E_1E_2 + 2\mathbb{P}_1^2 = S - m_1^2 - m_2^2$$

使う

$$B = \sqrt{(P_1\cdot P_2)^2 - m_1^2m_2^2}$$
$$= \sqrt{(P_{10}P_{20} - \mathbb{P}_1\cdot\mathbb{P}_2)^2 - m_1^2m_2^2}$$
$$= \sqrt{(E_1E_2 + \mathbb{P}_1^2)^2 - m_1^2m_2^2} = \sqrt{\frac{(S-m_1^2-m_2^2)^2}{4} - m_1^2m_2^2}$$
$$= \sqrt{\left[\frac{S-m_1^2-m_2^2}{2} - m_1m_2\right]\left[\frac{S-m_1^2-m_2^2}{2} + m_1m_2\right]}$$
$$= \sqrt{\frac{S-(m_1+m_2)^2}{2}\cdot\frac{S-(m_1-m_2)^2}{2}} = \frac{\sqrt{[S-(m_1-m_2)^2][S+(m_1+m_2)^2]}}{2} = \sqrt{S}\,|\mathbb{P}_i|$$

微分散乱断面積 を求めると、

$$\sigma = \frac{1}{4(2\pi)^2 v_{rel}E_1E_2}\underbrace{\int\frac{d^3q_1}{2E_{q_1}}\int\frac{d^3q_2}{2E_{q_2}}\delta(E_f - E_i)\delta^{(3)}(\mathbb{P}_f - \mathbb{P}_i)}_{I}$$
$$\times V^4\, 2E_1 2E_2\cdot 2E_{q_1}\cdot 2E_{q_2}\,|\langle f|T|i\rangle|^2$$

$I = \int d\Omega\,\frac{|\mathbb{P}_f|}{4\sqrt{S}}$

$v_{rel}E_1E_2 = \sqrt{S}\,|\mathbb{P}_i|$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(2\pi)^2}\cdot\frac{1}{\sqrt{S}\,|\mathbb{P}_i|}\cdot\frac{|\mathbb{P}_f|}{4\sqrt{S}}\cdot \sum_i V^4\cdot 2E_1 2E_2\cdot 2E_{q_1}\cdot 2E_{q_2}\,|\langle f|T|i\rangle|^2$$
$$= \frac{1}{(8\pi)^2 S}\cdot\frac{|\mathbb{P}_f|}{|\mathbb{P}_i|}\sum_{spin} V^4\cdot 2E_1\cdot 2E_2\cdot 2E_{q_1}\cdot 2E_{q_1}\,|\langle f|T|i\rangle|^2$$
$$= \frac{1}{(8\pi)^2 E_{Tot}^2}\cdot\frac{|\mathbb{P}_f|}{|\mathbb{P}_i|}\,|\langle f|t|i\rangle|^2$$

$\Rightarrow$ $|\langle f|t|i\rangle|^2 = \sum_{state} V^4\cdot 2E_1\cdot 2E_2\cdot 2E_{q_1}\cdot 2E_{q_2}\,|\langle f|T|i\rangle|^2$

粒子が変化しない(始状態と終状態で)、かつ弾性散乱の場合.

$|p_i| = |p_f|$ より.

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{1}{(8\pi)^2 E_{tot}^2} \cdot |\langle f|t|i\rangle|^2} ,$$

$\Downarrow$ 非相対論的極限

※ 全エネルギーについて非相対論的極限をとると.

$$E = \sqrt{p^2+m^2} = m\sqrt{1+\frac{p^2}{m^2}} \simeq m\left(1+\frac{p^2}{2m^2}\right)$$

$$= m + \frac{p^2}{2m} \simeq m$$

$$\therefore \underline{E_{tot} \simeq m_1+m_2}$$

換算質量 (reduced mass)

$$m_{red} = \frac{m_1 m_2}{m_1+m_2}$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{1}{(8\pi)^2 (m_1+m_2)^2} |M_{fi}|^2}$$

量子力学におけるBorn近似での散乱断面積

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \left|\frac{m_{red}}{2\pi}\int d^3x\, e^{i(k_i-k_f)\cdot x} V(x)\right|^2}$$

$$\frac{M_{fi}}{8\pi \cdot (m_1+m_2)} = \frac{m_{red}}{2\pi}\int d^3x\, e^{i(k_i-k_f)\cdot x} V(x)$$

$$\frac{M_{fi}}{4} = \frac{m_1 m_2}{\cancel{m_1+m_2}}\int d^3x\, e^{i(k_i-k_f)\cdot x} V(x)$$

$$\Leftrightarrow \boxed{V(x) = \frac{1}{4m_1m_2}\int \frac{d^3q}{(2\pi)^3} e^{iq\cdot x} M_{fi} , \quad q = p_f - p_i}$$

場の理論の散乱振幅からポテンシャルを計算する式.