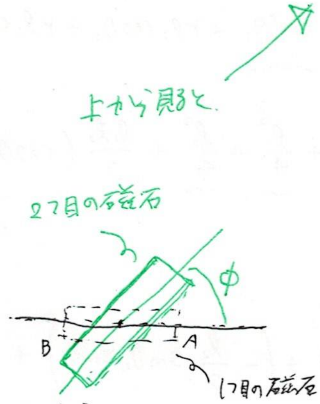
磁気モーメント

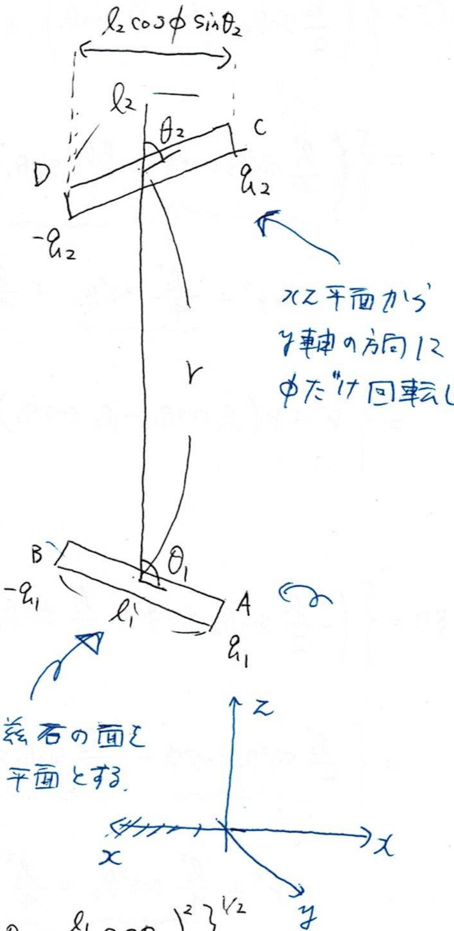
$$A\left(\frac{l_1}{2}\sin\theta_1,\ 0,\ \frac{l_1}{2}\cos\theta_1\right)$$

$$B\left(-\frac{l_1}{2}\sin\theta_1,\ 0,\ -\frac{l_1}{2}\cos\theta_1\right)$$

$$C\left(\frac{l_2}{2}\sin\theta_2\cos\phi,\ \frac{l_2}{2}\sin\theta_2\sin\phi,\ r+\frac{l_2}{2}\cos\theta_2\right)$$

$$D\left(-\frac{l_2}{2}\sin\theta_2\cos\phi,\ -\frac{l_2}{2}\sin\theta_2\sin\phi,\ r-\frac{l_2}{2}\cos\theta_2\right)$$





各点間の距離

$$AC=\left\{\left(\frac{l_2}{2}\sin\theta_2\cos\phi-\frac{l_1}{2}\sin\theta_1\right)^2+\left(\frac{l_2}{2}\sin\theta_2\sin\phi\right)^2+\left(r+\frac{l_2}{2}\cos\theta_2-\frac{l_1}{2}\cos\theta_1\right)^2\right\}^{1/2}$$

$$=\left\{\frac{l_2^2}{4}\sin^2\theta_2\cos^2\phi-\frac{l_1l_2}{2}\sin\theta_1\sin\theta_2\cos\phi+\frac{l_1^2}{4}\sin^2\theta_1+\frac{l_2^2}{4}\sin^2\theta_2\sin^2\phi\right.$$
$$\left.+r^2+\frac{l_2^2}{4}\cos^2\theta_2+\frac{l_1^2}{4}\cos^2\theta_1+rl_2\cos\theta_2-rl_1\cos\theta_1-\frac{l_1l_2}{2}\cos\theta_1\cos\theta_2\right\}^{1/2}$$

$$=\left\{r^2+r(l_2\cos\theta_2-l_1\cos\theta_1)+\frac{l_1^2}{4}+\frac{l_2^2}{4}-\frac{l_1l_2}{2}(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\right\}^{1/2}$$

$$AD=\left\{\left(-\frac{l_2}{2}\sin\theta_2\cos\phi-\frac{l_1}{2}\sin\theta_1\right)^2+\left(-\frac{l_2}{2}\sin\theta_2\sin\phi\right)^2+\left(r-\frac{l_2}{2}\cos\theta_2-\frac{l_1}{2}\cos\theta_1\right)^2\right\}^{1/2}$$

$$=\left\{\frac{l_2^2}{4}\sin^2\theta_2\cos^2\phi+\frac{l_1l_2}{2}\sin\theta_1\sin\theta_2\cos\phi+\frac{l_1^2}{4}\sin^2\theta_1+\frac{l_2^2}{4}\sin^2\theta_2\sin^2\phi\right.$$
$$\left.+r^2+\frac{l_2^2}{4}\cos^2\theta_2+\frac{l_1^2}{4}\cos^2\theta_1-rl_2\cos\theta_2-rl_1\cos\theta_1+\frac{l_1l_2}{2}\cos\theta_1\cos\theta_2\right\}^{1/2}$$

$$=\left\{r^2-r(l_1\cos\theta_1+l_2\cos\theta_2)+\frac{l_1^2}{4}+\frac{l_2^2}{4}+\frac{l_1l_2}{2}(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\right\}^{1/2}$$

$$BC = \left\{ \left(\frac{l_2}{2}\sin\theta_2\cos\phi + \frac{l_1}{2}\sin\theta_1 \right)^2 + \left(\frac{l_2}{2}\sin\theta_2\sin\phi \right)^2 + \left(r + \frac{l_2}{2}\cos\theta_2 + \frac{l_1}{2}\cos\theta_1 \right)^2 \right\}^{1/2}$$

$$= \left\{ \frac{l_2^2}{4}\sin^2\theta_2\cos^2\phi + \frac{l_1 l_2}{2}\sin\theta_1\sin\theta_2\cos\phi + \frac{l_1^2}{4}\sin^2\theta_1 + \frac{l_2^2}{4}\sin^2\theta_2\sin^2\phi \right.$$

$$\left. + r^2 + \frac{l_2^2}{4}\cos^2\theta_2 + \frac{l_1^2}{4}\cos^2\theta_1 + r l_2\cos\theta_2 + r l_1\cos\theta_1 + \frac{l_1 l_2}{2}\cos\theta_1\cos\theta_2 \right\}^{1/2}$$

$$= \left\{ r^2 + r(l_1\cos\theta_1 + l_2\cos\theta_2) + \frac{l_1^2}{4} + \frac{l_2^2}{4} + \frac{l_1 l_2}{2}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi) \right\}^{1/2}$$

$$BD = \left\{ \left(-\frac{l_2}{2}\sin\theta_2\cos\phi + \frac{l_1}{2}\sin\theta_1 \right)^2 + \left(-\frac{l_2}{2}\sin\theta_2\sin\phi \right)^2 + \left(r - \frac{l_2}{2}\cos\theta_2 + \frac{l_1}{2}\cos\theta_1 \right)^2 \right\}^{1/2}$$

$$= \left\{ \frac{l_2^2}{4}\sin^2\theta_2\cos^2\phi - \frac{l_1 l_2}{2}\sin\theta_1\sin\theta_2\cos\phi + \frac{l_1^2}{4}\sin^2\theta_1 + \frac{l_2^2}{4}\sin^2\theta_2\sin^2\phi \right.$$

$$\left. + r^2 + \frac{l_2^2}{4}\cos^2\theta_2 + \frac{l_1^2}{4}\cos^2\theta_1 - r l_2\cos\theta_2 + r l_1\cos\theta_1 - \frac{l_1 l_2}{2}\cos\theta_1\cos\theta_2 \right\}^{1/2}$$

$$= \left\{ r^2 + r(l_1\cos\theta_1 - l_2\cos\theta_2) + \frac{l_1^2}{4} + \frac{l_2^2}{4} - \frac{l_1 l_2}{2}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi) \right\}^{1/2}$$

磁極の正負に注意して、エネルギーを求める。

$$U = \frac{q_{m_1} q_{m_2}}{4\pi\mu_0}\left(\frac{1}{AC} + \frac{1}{BD} - \frac{1}{AD} - \frac{1}{BC}\right)$$

それぞれ、$\frac{l^2}{r^2}$ のオーダーまで展開する。

- $f = (1+x)^{-1/2}$ の展開
- $f' = -\frac{1}{2}(1+x)^{-3/2}$
- $f'' = \frac{3}{4}(1+x)^{-5/2}$ より
- $f \simeq 1 - \frac{1}{2}x + \frac{3}{8}x^2$

$$AC^{-1} = r^{-1}\left\{1 + \frac{l_2}{r}\cos\theta_2 - \frac{l_1}{r}\cos\theta_1 + \frac{l_1^2}{4r^2} + \frac{l_2^2}{4r^2} - \frac{l_1 l_2}{2r^2}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi)\right\}^{-1/2}$$

$$\simeq \frac{1}{r}\left\{1 - \frac{l_2}{2r}\cos\theta_2 + \frac{l_1}{2r}\cos\theta_1 - \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} + \frac{l_1 l_2}{4r^2}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi) + \frac{3}{8}\left[\frac{l_2}{r}\cos\theta_2 - \frac{l_1}{r}\cos\theta_1 + \frac{l_1^2}{4r^2} + \frac{l_2^2}{4r^2} - \frac{l_1 l_2}{2r^2}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi)\right]^2\right\}$$

$$\simeq \frac{1}{r}\left\{1 - \frac{l_2}{2r}\cos\theta_2 + \frac{l_1}{2r}\cos\theta_1 - \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} + \frac{l_1 l_2}{4r^2}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi) + \frac{3}{8}\left(\frac{l_2}{r}\cos\theta_2 - \frac{l_1}{r}\cos\theta_1\right)^2\right\}$$

$$= \frac{1}{r}\left\{1 - \frac{l_2}{2r}\cos\theta_2 + \frac{l_1}{2r}\cos\theta_1 - \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} + \frac{l_1 l_2}{4r^2}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi) + \frac{3}{8}\frac{l_2^2}{r^2}\cos^2\theta_2 - \frac{3}{4}\frac{l_1 l_2}{r^2}\cos\theta_1\cos\theta_2 + \frac{3}{8}\frac{l_1^2}{r^2}\cos^2\theta_1\right\}$$

$$= \frac{1}{r}\left\{1 + \frac{l_1}{2r}\cos\theta_1 - \frac{l_2}{2r}\cos\theta_2 - \frac{l_1^2}{8r^2}(1 - 3\cos^2\theta_1) - \frac{l_2^2}{8r^2}(1 - 3\cos^2\theta_2) + \frac{l_1 l_2}{4r^2}(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi)\right\}$$

$$AD^{-1} = r^{-1}\left\{1 - \frac{l_1}{r}\cos\theta_1 - \frac{l_2}{r}\cos\theta_2 + \frac{l_1^2}{4r^2} + \frac{l_2^2}{4r^2} + \frac{l_1 l_2}{2r^2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)\right\}^{-1/2}$$

$$\simeq \frac{1}{r}\left\{1 + \frac{l_1}{2r}\cos\theta_1 + \frac{l_2}{2r}\cos\theta_2 - \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} - \frac{l_1 l_2}{4r^2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)\right.$$

$$\left. + \frac{3}{8}\left(-\frac{l_1}{r}\cos\theta_1 - \frac{l_2}{r}\cos\theta_2\right)^2\right\}$$

$$= \frac{1}{r}\left\{1 + \frac{l_1}{2r}\cos\theta_1 + \frac{l_2}{2r}\cos\theta_2 - \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} - \frac{l_1 l_2}{4r^2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)\right.$$

$$\left. + \frac{3}{8}\frac{l_1^2}{r^2}\cos^2\theta_1 + \frac{3}{4}\frac{l_1 l_2}{r^2}\cos\theta_1\cos\theta_2 + \frac{3}{8}\frac{l_2^2}{r^2}\cos^2\theta_2\right\}$$

$$= \frac{1}{r}\left\{1 + \frac{l_1}{2r}\cos\theta_1 + \frac{l_2}{2r}\cos\theta_2 - \frac{l_1^2}{8r^2}\left(1 - 3\cos^2\theta_1\right) - \frac{l_2^2}{8r^3}\left(1 - 3\cos^2\theta_2\right) - \frac{l_1 l_2}{4r^2}\left(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)\right\}$$

$$BC^{-1} = r^{-1}\left\{1 + \frac{l_1}{r}\cos\theta_1 + \frac{l_2}{r}\cos\theta_2 + \frac{l_1^2}{4r^2} + \frac{l_2^2}{4r^2} + \frac{l_1 l_2}{2r^2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)\right\}^{-1/2}$$

$$\simeq \frac{1}{r}\left\{1 - \frac{l_1}{2r}\cos\theta_1 - \frac{l_2}{2r}\cos\theta_2 - \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} - \frac{l_1 l_2}{4r^2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)\right.$$

$$\left. + \frac{3}{8}\left(\frac{l_1}{r}\cos\theta_1 + \frac{l_2}{r}\cos\theta_2\right)^2\right\}$$

$$= \frac{1}{r}\left\{1 - \frac{l_1}{2r}\cos\theta_1 - \frac{l_2}{2r}\cos\theta_2 - \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} - \frac{l_1 l_2}{4r^2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)\right.$$

$$\left. + \frac{3}{8}\frac{l_1^2}{r^2}\cos^2\theta_1 + \frac{3}{4}\frac{l_1 l_2}{r^2}\cos\theta_1\cos\theta_2 + \frac{3}{8}\frac{l_2^2}{r^2}\cos^2\theta_2\right\}$$

$$= \frac{1}{r}\left\{1 - \frac{l_1}{2r}\cos\theta_1 - \frac{l_2}{2r}\cos\theta_2 - \frac{l_1^2}{8r^2}\left(1 - 3\cos^2\theta_1\right) - \frac{l_2^2}{8r^2}\left(1 - 3\cos^2\theta_2\right) - \frac{l_1 l_2}{4r^2}\left(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)\right\}$$

$$BD^{-1} = r^{-1}\left\{ 1+\frac{l_1}{r}\cos\theta_1 - \frac{l_2}{r}\cos\theta_2 + \frac{l_1^2}{4r^2} + \frac{l_2^2}{4r^2} - \frac{l_1 l_2}{2r^2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)\right\}^{-1/2}$$

$$\simeq \frac{1}{r}\left\{ 1-\frac{l_1}{2r}\cos\theta_1 + \frac{l_2}{2r}\cos\theta_2 - \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} + \frac{l_1 l_2}{4r^2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right) + \frac{3}{8}\left(\frac{l_1}{r}\cos\theta_1 - \frac{l_2}{r}\cos\theta_2\right)^2 \right\}$$

$$= \frac{1}{r}\left\{ 1-\frac{l_1}{2r}\cos\theta_1 + \frac{l_2}{2r}\cos\theta_2 - \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} + \frac{l_1 l_2}{4r^2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right) + \frac{3}{8}\frac{l_1^2}{r^2}\cos^2\theta_1 - \frac{3}{4}\frac{l_1 l_2}{r^2}\cos\theta_1\cos\theta_2 + \frac{3}{8}\frac{l_2^2}{r^2}\cos^2\theta_2 \right\}$$

$$= \frac{1}{r}\left\{ 1-\frac{l_1}{2r}\cos\theta_1 + \frac{l_2}{2r}\cos\theta_2 \cancel{- \frac{l_1^2}{8r^2} - \frac{l_2^2}{8r^2} +} - \frac{l_1^2}{8r^2}\left(1-3\cos^2\theta_1\right) - \frac{l_2^2}{8r^2}\left(1-3\cos^2\theta_2\right) + \frac{l_1 l_2}{4r^2}\left(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right) \right\}$$

$$\Downarrow$$

$$\frac{1}{AC} + \frac{1}{BD} - \frac{1}{AD} - \frac{1}{BC}$$

$$= \frac{1}{r}\Big\{ \cancel{1} + \cancel{\frac{l_1}{2r}\cos\theta_1} - \cancel{\frac{l_2}{2r}\cos\theta_2} - \frac{l_1^2}{8r^2}\left(1-3\cos^2\theta_1\right) - \frac{l_2^2}{8r^2}\left(1-3\cos^2\theta_2\right) + \frac{l_1 l_2}{4r^2}\left(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)$$

$$+ \cancel{1} - \cancel{\frac{l_1}{2r}\cos\theta_1} + \cancel{\frac{l_2}{2r}\cos\theta_2} - \frac{l_1^2}{8r^2}\left(1-3\cos^2\theta_1\right) - \frac{l_2^2}{8r^2}\left(1-3\cos^2\theta_2\right) + \frac{l_1 l_2}{4r^2}\left(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)$$

$$- \cancel{1} - \cancel{\frac{l_1}{2r}\cos\theta_1} - \cancel{\frac{l_2}{2r}\cos\theta_2} + \frac{l_1^2}{8r^2}\left(1-3\cos^2\theta_1\right) + \frac{l_2^2}{8r^2}\left(1-3\cos^2\theta_2\right) + \frac{l_1 l_2}{4r^2}\left(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)$$

$$- \cancel{1} + \cancel{\frac{l_1}{2r}\cos\theta_1} + \cancel{\frac{l_2}{2r}\cos\theta_2} + \frac{l_1^2}{8r^2}\left(1-3\cos^2\theta_1\right) + \frac{l_2^2}{8r^2}\left(1-3\cos^2\theta_2\right) + \frac{l_1 l_2}{4r^2}\left(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right) \Big\}$$

$$= \frac{l_1 l_2}{r^3}\left(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)$$

$$\therefore\ V = \frac{q_{m_1} q_{m_2} l_1 l_2}{4\pi\mu_0 r^3}\left(-2\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi\right)$$

各磁気モーメントは、

$$\mathbb{m}_1 = q_{m_1}(\vec{OA} - \vec{OB}) = q_{m_1}\vec{BA} \qquad \vec{BA} = (l_1 \sin\theta_1, 0, l_1 \cos\theta_1)$$

$$\mathbb{m}_2 = q_{m_2}(\vec{OC} - \vec{OD}) = q_{m_2}\vec{DC} \qquad \vec{DC} = (l_2 \sin\theta_2 \cos\phi, l_2 \sin\theta_2 \sin\phi, l_2 \cos\theta_2)$$

$$\mathbb{m}_1 \cdot \mathbb{m}_2 = q_{m_1} q_{m_2} (l_1 l_2 \sin\theta_1 \sin\theta_2 \cos\phi + l_1 l_2 \cos\theta_1 \cos\theta_2)$$

$$= q_{m_1} q_{m_2} l_1 l_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi)$$

また、$\mathbb{m}_1 \cdot \mathbb{r} = q_{m_1}(0, 0, r l_1 \cos\theta_1)$

$\mathbb{m}_2 \cdot \mathbb{r} = q_{m_2}(0, 0, r l_2 \cos\theta_2)$ $\Rightarrow$

$\mathbb{m}_1 \cdot \mathbb{r} = r l_1 \cos\theta_1 q_{m_1}$

$\mathbb{m}_2 \cdot \mathbb{r} = r l_2 \cos\theta_2 q_{m_2}$ $\Rightarrow$ $(\mathbb{m}_1 \cdot \mathbb{r})(\mathbb{m}_2 \cdot \mathbb{r}) = r^2 l_1 l_2 \cos\theta_1 \cos\theta_2 \times q_{m_1} q_{m_2}$

したがって、

$$U = \frac{q_{m_1} q_{m_2}}{4\pi\mu_0 r}\left[\frac{l_1 l_2}{r^2}\left\{(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi) - 3\cos\theta_1 \cos\theta_2\right\}\right]$$

$$= \frac{1}{4\pi\mu_0 r^3}\left[\mathbb{m}_1 \cdot \mathbb{m}_2 - 3\frac{(\mathbb{m}_1 \cdot \mathbb{r})(\mathbb{m}_2 \cdot \mathbb{r})}{r^2}\right]$$