Lorentz変換の下でのMaxwell方程式の変化.

Lorentz 変換 (x-方向)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\frac{d}{d\underset{c}{t}} = \frac{dt'}{dt}\frac{d}{dct'} + \frac{dx'}{d(ct)}\frac{d}{dx'} + \underbrace{\frac{dy'}{d(ct)}}_{0}\frac{d}{dy'} + \underbrace{\frac{dz'}{d(ct)}}_{0}\frac{d}{dz'}$$

$$\downarrow$$

$$\partial_t = \gamma\partial_{t'} - \gamma\beta\partial_{x'} \quad \cdots ①$$

$$\partial_x = \frac{d(ct')}{dx}\frac{d}{d(ct')} + \frac{dx'}{dx}\frac{d}{dx'} + \underbrace{\frac{dy'}{dx}}_{0}\frac{d}{dy'} + \underbrace{\frac{dz'}{dx}}_{0}\frac{d}{dz'}$$

$$= -\gamma\beta\partial_{t'} + \gamma\partial_{x'} \quad \cdots ②$$

$$\partial_y = \underbrace{\frac{d(ct')}{dy}}_{0}\frac{d}{d(ct')} + \underbrace{\frac{dx'}{dy}}_{0}\frac{d}{dx'} + \underbrace{\frac{dy'}{dy}}_{1}\frac{d}{dy'} + \underbrace{\frac{dz'}{dy}}_{0}\frac{d}{dz'} = \partial_{y'}$$

zも同様に、$\partial_z = \partial_{z'}$

$$\Rightarrow \boxed{\begin{aligned} &\partial_t = \gamma\partial_{t'} - \gamma\beta\partial_{x'} \\ &\partial_x = -\gamma\beta\partial_{t'} + \gamma\partial_{x'} \\ &\partial_{y'} = \partial_y \\ &\partial_{z'} = \partial_z \end{aligned}}$$

Maxwell 方程式

$$\nabla\cdot\mathbb{E}=\rho,\quad \nabla\cdot\mathbb{B}=0,$$

$$\nabla\times\mathbb{E}+\frac{d\mathbb{B}}{dt}=0,\quad \nabla\times\mathbb{B}=\mathbb{J}+\frac{d\mathbb{E}}{dt}$$

$\boxed{\text{Gauss の法則}}$

$$\nabla\cdot\mathbb{E}=(-\gamma\beta\partial_{t'}+\gamma\partial_{x'})E_x+\partial_y E_y+\partial_z E_z=\rho$$

$\boxed{\text{単磁荷なしの法則}}$

$$\nabla\cdot\mathbb{B}=(-\gamma\beta\partial_{t'}+\gamma\partial_{x'})B_x+\partial_y B_y+\partial_z B_z=0.$$

$\boxed{\text{Faraday の電磁誘導の法則}}$

$$\gamma\partial_{x'}B_x=\gamma\beta\partial_{t'}B_x-\partial_{y'}B_y-\partial_{z'}B_z$$

イコール

x-成分

$$\partial_y E_z-\partial_z E_y+\partial_t B_x=0$$

$$\Leftrightarrow \partial_{y'}E_z-\partial_{z'}E_y+\gamma\partial_{t'}B_x-\gamma\beta\partial_{x'}B_x=0$$

$$\Leftrightarrow \underline{\partial_{y'}E_z}-\underline{\partial_{z'}E_y}+\underline{\gamma\partial_{t'}B_x}-(\underline{\gamma\beta^2\partial_{t'}B_x}-\underline{\beta\partial_{y'}B_y}-\underline{\beta\partial_{z'}B_z})=0$$

$$\Leftrightarrow \partial_{y'}(E_z+\beta B_y)-\partial_{z'}(E_y-\beta B_z)+\gamma(1-\beta^2)\partial_{t'}B_x=0$$

$$\Leftrightarrow \partial_{y'}(E_z+\beta B_y)-\partial_{z'}(E_y-\beta B_z)+\frac{1}{\gamma}\partial_{t'}B_x=0$$

y成分

$$\partial_{z'}E_x - (-\gamma\beta\partial_{t'} + \gamma\partial_{x'})E_z + (\gamma\partial_{t'} - \gamma\beta\partial_{x'})B_y = 0$$

$$\Leftrightarrow \partial_{z'}E_x - \gamma\partial_{x'}(E_z + \beta B_y) + \gamma\partial_{t'}(B_y + \beta E_z) = 0$$

z成分

$$(-\gamma\beta\partial_{t'} + \gamma\partial_{x'})E_y - \partial_{y'}E_x + (\gamma\partial_{t'} - \gamma\beta\partial_{x'})B_z = 0$$

$$\Leftrightarrow \gamma\partial_{x'}(E_y - \beta B_z) - \partial_{y'}E_x + \gamma\partial_{t'}(B_z - \beta E_y) = 0$$

Ampare - Maxwellの法則

x成分

ガウスの法則より

$$\gamma\partial_{x'}E_x = \gamma\beta\partial_{t'}E_x - \partial_{y'}E_y - \partial_{z'}E_z + \rho$$

代入

$$\partial_{y'}B_z - \partial_{z'}B_y = J_x + (\gamma\partial_{t'} - \gamma\beta\partial_{x'})E_x$$

$$= J_x + \gamma\partial_{t'}E_x - \gamma\beta^2\partial_{t'}E_x + \beta\partial_{y'}E_y + \beta\partial_{z'}E_z - \beta\rho$$

$$\partial_{y'}(B_z - \beta E_y) - \partial_{z'}(B_y + \beta E_z) = J_x - \beta\rho + \gamma(1-\beta^2)\partial_{t'}E_x$$

$$= J_x - \beta\rho + \frac{1}{\gamma}\partial_{t'}E_x$$

y成分

$$\partial_{z'}B_x - (-\gamma\beta\partial_{t'} + \gamma\partial_{x'})B_z = J_y + (\gamma\partial_{t'} - \gamma\beta\partial_{x'})E_y$$

$$\partial_{z'}B_x - \gamma\partial_{x'}(B_z - \beta E_y) = J_y + \gamma\partial_{t'}(E_y - \beta B_z)$$

z成分

$$(-\gamma\beta\partial_{t'} + \gamma\partial_{x'})B_y - \partial_{y'}B_x = J_z + (\gamma\partial_{t'} - \gamma\beta\partial_{x'})E_z$$

$$\gamma\partial_{x'}(B_y + \beta E_z) - \partial_{y'}B_x = J_z + \gamma\partial_{t'}(E_z + \beta B_y)$$

連続の方程式

$$\frac{d\rho}{dt} + \nabla \cdot J = 0 \quad \text{より}$$

$$(\gamma\partial_{t'} - \gamma\beta\partial_{x'})\rho + \gamma(-\beta\partial_{t'} + \partial_{x'})J_x + \partial_{y'}J_y + \partial_{z'}J_z = 0$$

$$\Leftrightarrow \gamma\partial_{t'}(\rho - \beta J_x) + \gamma\partial_{x'}(J_x - \beta\rho) + \partial_{y'}J_y + \partial_{z'}J_z = 0$$

これにより、$\boxed{J_x' = \gamma(J_x - \beta\rho)}$, $\boxed{\rho' = \gamma(\rho - \beta J_x)}$ に変換されると分かる。

まとめると、

Faradayの法則

$$\partial_{y'}[\gamma(E_z + \beta B_y)] - \partial_{z'}[\gamma(E_y - \beta B_z)] + \partial_{t'}B_x = 0$$

$$\partial_{z'}E_x - \partial_{x'}[\gamma(E_z + \beta B_y)] + \partial_{t'}[\gamma(B_y + \beta E_z)] = 0$$

$$\partial_{x'}[\gamma(E_y - \beta B_z)] - \partial_{y'}E_x + \partial_{t'}[\gamma(B_z - \beta E_y)] = 0$$

Ampere-Maxwellの法則

$$\partial_{y'}[\gamma(B_z - \beta E_y)] - \partial_{z'}[\gamma(B_y + \beta E_z)] = \gamma(J_x - \beta\rho) + \partial_{t'}E_x$$

$$\partial_{z'}B_x - \partial_{x'}[\gamma(B_z - \beta E_y)] = J_y + \partial_{t'}[\gamma(E_y - \beta B_z)]$$

$$\partial_{x'}[\gamma(B_y + \beta E_z)] - \partial_{y'}B_x = J_z + \partial_{t'}[\gamma(E_z + \beta B_y)]$$

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$$\boxed{\begin{aligned} &E_{x'} = E_x, \quad E_{y'} = \gamma(E_y - \beta B_z), \quad E_{z'} = \gamma(E_z + \beta B_y) \\ &B_{x'} = B_x, \quad B_{y'} = \gamma(B_y + \beta E_z), \quad B_{z'} = \gamma(B_z - \beta E_y) \\ &J_{x'} = \gamma(J_x - \beta \rho) \quad \rho' = \gamma(\rho - \beta J_x) \end{aligned}}$$

とおきなおす。

$\Downarrow$

$$\left.\begin{aligned} &\partial_{y'} E_{z'} - \partial_{z'} E_{y'} + \partial_{t'} B_{x'} = 0 \\ &\partial_{z'} E_{x'} - \partial_{x'} E_{z'} + \partial_{t'} B_{y'} = 0 \\ &\partial_{x'} E_{y'} - \partial_{y'} E_{x'} + \partial_{t'} B_{z'} = 0 \end{aligned}\right\} \quad \boxed{\nabla' \times \mathbb{E}' + \frac{d\mathbb{B}'}{dt} = 0}$$

$$\left.\begin{aligned} &\partial_{y'} B_{z'} - \partial_{z'} B_{y'} = J_{x'} + \partial_{t'} E_{x'} \\ &\partial_{z'} B_{x'} - \partial_{x'} B_{z'} = J_y + \partial_{t'} E_{y'} \\ &\partial_{x'} B_{y'} - \partial_{y'} B_{x'} = J_{z'} + \partial_{t'} E_{z'} \end{aligned}\right\} \quad \boxed{\nabla' \times \mathbb{B}' = \mathbb{J}' + \frac{d\mathbb{E}'}{dt'}}$$

Gaussの法則の変化

$$\gamma \partial_{x'} E_{x'} + \partial_{y'} E_y + \partial_{z'} E_z - \gamma\beta \partial_{t'} E_x = \rho \quad \xleftarrow[\text{代入}]{} \quad \partial_{y'} B_{z'} - \partial_{z'} B_{y'} = J_{x'} + \partial_{t'} E_{x'}$$

$$\gamma \partial_{x'} E_{x'} + \partial_{y'} E_y + \partial_{z'} E_z - \gamma\beta(\partial_{y'} B_{z'} - \partial_{z'} B_{y'} - J_{x'}) = \rho$$

$$\Leftrightarrow \gamma \partial_{x'} E_{x'} + \underline{\partial_{y'} E_y} + \underline{\partial_{z'} E_z} - \gamma^2\beta \partial_{y'}(B_z - \underline{\beta E_y}) + \gamma^2\beta \partial_{z'}(B_y + \underline{\beta E_z}) + \gamma^2\beta(J_x - \underline{\beta\rho}) = \underline{\rho}$$

$$\Leftrightarrow \gamma \partial_{x'} E_{x'} + \underline{\gamma^2 \partial_{y'} E_y} + \underline{\gamma^2 \partial_{z'} E_z} - \underline{\gamma^2\beta \partial_{y'} B_z} + \underline{\gamma^2\beta \partial_{z'} B_y} + \gamma^2\beta J_x = \gamma^2 \rho$$

$$1 + \gamma^2\beta^2 = 1 + \frac{\beta^2}{1-\beta^2} = \frac{1}{1-\beta^2} = \gamma^2$$

$$\Leftrightarrow \gamma \partial_{x'} E_{x'} + \gamma \partial_{y'} E_{y'} + \gamma \partial_{z'} E_{z'} = \gamma \rho'$$

$$\Leftrightarrow \partial_{x'} E_{x'} + \partial_{y'} E_{y'} + \partial_{z'} E_{z'} = \rho'$$

$$\therefore \boxed{\nabla' \cdot \mathbb{E}' = \rho'}$$

単磁荷なしの法則

$$\gamma\partial_{x'}B_{x'} + \partial_{y'}B_y + \partial_{z'}B_z - \gamma\beta\partial_{t'}B_x = 0. \quad \xleftarrow{\text{代入}} \quad \partial_{y'}E_{z'} - \partial_{z'}E_{y'} + \partial_{t'}B_{x'} = 0$$

$$\Leftrightarrow \gamma\partial_{x'}B_{x'} + \partial_{y'}B_y + \partial_{z'}B_z - \gamma\beta(-\partial_{y'}E_{z'} + \partial_{z'}E_{y'}) = 0$$

$$\Leftrightarrow \gamma\partial_{x'}B_{x'} + \partial_{y'}B_y + \partial_{z'}B_z + \gamma^2\beta\partial_{y'}(E_z + \beta B_y) - \gamma^2\beta\partial_{z'}(E_y - \beta B_z) = 0$$

$$\Leftrightarrow \gamma\partial_{x'}B_{x'} + \gamma^2\partial_{y'}B_y + \gamma^2\partial_{z'}B_z + \gamma^2\beta\partial_{y'}E_z - \gamma^2\beta\partial_{z'}E_y = 0$$

$$\Leftrightarrow \gamma\partial_{x'}B_{x'} + \gamma^2\partial_{y'}(B_y + \beta E_z) + \gamma^2\partial_{z'}(B_z - \beta E_y) = 0$$

$$\Leftrightarrow \partial_{x'}B_{x'} + \partial_{y'}B_{y'} + \partial_{z'}B_{z'} = 0$$

$$\therefore \boxed{\nabla' \cdot \mathbb{B}' = 0}$$

以上より、Maxwell方程式はLorentz変換の下で不変。

Lorentz変換の下でのMaxwell方程式.

4元の形式ではより簡単な形となる。

$$\partial_\nu F^{\nu\mu} = J^\mu$$ … Maxwell方程式

$\Downarrow$ Lorentz変換

$$L_\nu{}^\lambda \partial_\lambda L^\nu{}_\sigma L^\mu{}_\varepsilon F^{\sigma\varepsilon} = L^\mu{}_\varepsilon J^\varepsilon$$

共通なので逆行列をかけて落ちる.

$$\Leftrightarrow \underbrace{L_\nu{}^\lambda L^\nu{}_\sigma}_{g^\lambda{}_\sigma} \partial_\lambda F^{\sigma\mu} = J^\mu$$

$\Leftrightarrow \partial_\lambda F^{\lambda\mu} = J^\mu$ となり、Lorentz変換の下でMaxwell方程式は不変.