Foldy - Wouthuysen transformation と non-relativistic limit of the Dirac eq.

FW transformation は、フェルミオンの波動関数のユニタリ変換。

$$\psi \to \psi' = U\psi$$

ユニタリ演算子 Uは 4×4 行列で、

$\boxed{U = e^{\beta\alpha\cdot\hat{p}\theta}}$ と書かれる。 $\hat{p} = \frac{p}{|p|}$

ここで、

$$(\beta\alpha\cdot\hat{p})^2 = (\gamma^i\hat{p}^i)^2 = \gamma^i\gamma^j\hat{p}^i\hat{p}^j$$

$$= \frac{1}{2}(\gamma^i\gamma^j + \gamma^j\gamma^i)\hat{p}^i\hat{p}^j = -g^{ij}\hat{p}^i\hat{p}_j = -1.$$

であるから、

$$U = 1 + \beta\alpha\cdot\hat{p}\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!}\beta\alpha\cdot\hat{p} + \cdots$$

$$= \cos\theta\cdot I_4 + \beta\alpha\cdot\hat{p}\sin\theta. = I_4\cos\theta + \gamma\cdot\hat{p}\sin\theta.$$

この inverse については、

$$U^{-1} = e^{-\beta\alpha\cdot\hat{p}\theta} = I_4\cos\theta - \beta\alpha\cdot\hat{p}\sin\theta.$$

このとき、 $\exp[-\beta\alpha\cdot\hat{p}\theta]\beta\exp[\beta\alpha\cdot\hat{p}\theta]$

$$= (I_4\cos\theta - \beta(\alpha\cdot\hat{p})\sin\theta)\beta(I_4\cos\theta + \beta(\alpha\cdot\hat{p})\sin\theta)$$

$$= \beta\cos^2\theta - \beta(\alpha\cdot\hat{p})\sin\theta\beta\beta(\alpha\cdot\hat{p})\sin\theta - \beta(\alpha\cdot\hat{p})\sin\theta\beta\cos\theta + \cos\theta\beta\beta(\alpha\cdot\hat{p})\sin\theta$$

$$= \beta(\cos^2\theta - \sin^2\theta) + 2(\alpha\cdot\hat{p})\sin\theta\cos\theta$$

$$= (\alpha\cdot\hat{p})\sin2\theta + \beta\cos2\theta$$

よって、Dirac方程式 $(\alpha\cdot p + \beta m)u = Eu$ に対して

$$\alpha\cdot p + \beta m = \sqrt{p^2+m^2}\left(\alpha\cdot\hat{p}\frac{|p|}{\sqrt{p^2+m^2}} + \beta\frac{m}{\sqrt{|p|^2+m^2}}\right) = \sqrt{p^2+m^2}\,(\alpha\cdot\hat{p}\sin2\theta + \beta\cos2\theta)$$

$\tan2\theta = \frac{|p|}{m}$

とすると、 $U^{-1}\sqrt{p^2+m^2}\,\beta U = H(p)$

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Dirac Hamiltonian

$$H_D = \alpha\cdot p + \beta m$$

この変換は、

$$H_D' = UH_DU^{-1} = (\cos\theta + \gamma^i \hat{p}^i \sin\theta)\,\gamma^0(\gamma^j p^j + m)(\cos\theta - \gamma^k \hat{p}^k \sin\theta)$$

$$= \gamma^0(\gamma^j p^j + m)\cos\theta(\cos\theta - \gamma^k \hat{p}^k \sin\theta)$$

$$- \gamma^0 \gamma^i \hat{p}^i \sin\theta(\gamma^j p^j + m)(\cos\theta - \gamma^k \hat{p}^k \sin\theta)$$

$$= (\alpha\cdot p + \beta m)\cos\theta(\cos\theta - \beta\alpha\cdot\hat{p}\sin\theta)$$

$$- (\alpha\cdot p + \beta m)\beta\alpha\cdot\hat{p}\sin\theta(\cos\theta - \beta\alpha\cdot\hat{p}\sin\theta)$$

$$= (\alpha\cdot p + \beta m)(\cos\theta - \beta\alpha\cdot\hat{p}\sin\theta)^2$$

$$= (\alpha\cdot p + \beta m)\exp(-2\beta\alpha\cdot\hat{p}\theta)$$

$$= (\alpha\cdot p + \beta m)(\cos 2\theta - \beta\alpha\cdot\hat{p}\sin 2\theta)$$

$$= (\alpha\cdot p)\left(\cos 2\theta - \frac{m}{|p|}\sin 2\theta\right) + \beta(m\cos 2\theta + |p|\sin 2\theta)$$

$(\alpha\cdot p)$ 部分を消去するために、$\cos 2\theta - \frac{m}{|p|}\sin 2\theta = 0$ すなわち $\tan 2\theta = \frac{|p|}{m}$ となるように選ぶと、

$$\therefore\ \cos 2\theta = \sqrt{\frac{1}{1+\frac{p^2}{m^2}}} = \sqrt{\frac{m^2}{m^2+p^2}} = \frac{m}{\sqrt{m^2+p^2}},\qquad \sin 2\theta = \frac{|p|}{\sqrt{m^2+p^2}}$$

よって $H_D' = \beta\left(\frac{m^2}{\sqrt{m^2+p^2}} + \frac{p^2}{\sqrt{m^2+p^2}}\right) = \beta\sqrt{m^2+p^2}$

$|p| \ll m$ のとき、

$$H_D' \simeq \beta m\left(1 + \frac{p^2}{2m^2}\right) = \beta\left(m + \frac{p^2}{2m}\right)$$

Non-relativistic limit of the Dirac Hamiltonian in general form.

$$H_D = c\alpha\cdot\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) + mc^2\beta + e\phi$$

Dirac方程式 $i\hbar\frac{\partial\psi}{\partial t} = H_D\psi$ に ユニタリ変換 $\psi' = e^{iS}\psi$ 、

$$\text{左辺} = i\hbar\frac{\partial}{\partial t}\left(e^{-iS}\psi'\right) = i\hbar\left(\frac{\partial}{\partial t}e^{-iS}\right)\psi' + e^{-iS}\left(i\hbar\frac{\partial\psi'}{\partial t}\right)$$

$$\text{右辺} = H_D e^{-iS}\psi'$$

よって $$e^{-iS}\left(i\hbar\frac{\partial\psi'}{\partial t}\right) = \left(H_D e^{-iS} - i\hbar\left(\frac{\partial}{\partial t}e^{-iS}\right)\right)\psi'$$

$$\therefore\ i\hbar\frac{\partial\psi'}{\partial t} = \left[e^{iS}\left(H_D - i\hbar\frac{\partial}{\partial t}\right)e^{-iS}\right]\psi' = H_D'\psi'$$

ベーカー・ハウスドルフの定理より $e^{\lambda A}Be^{-\lambda A} = B + \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \cdots$ ｜ オーダーは4次までとる。

$$e^{iS}H_De^{-iS} = H_D + i[S,H_D] - \frac{1}{2}[S,[S,H_D]] - \frac{i}{6}[S,[S,[S,H_D]]] + \frac{1}{24}[S,[S,[S,[S,H_D]]]] + \cdots$$

また、$$e^{iS}\left(\frac{\partial}{\partial t}e^{-iS}\right) = e^{iS}\left\{-i\dot{S} - \frac{1}{2}(\dot{S}S + S\dot{S}) + \frac{i}{6}(\dot{S}S^2 + S\dot{S}S + S^2\dot{S}) + \frac{1}{24}(\dot{S}S^3 + S\dot{S}S^2 + S^2\dot{S}S + S^3\dot{S})\right\}$$

$$= \left(1 + iS - \frac{1}{2}S^2 - \frac{i}{6}S^3\right)\left\{-i\dot{S} - \frac{1}{2}(\dot{S}S + S\dot{S}) + \frac{i}{6}(\dot{S}S^2 + S\dot{S}S + S^2\dot{S}) + \frac{1}{24}(\dot{S}S^3 + S\dot{S}S^2 + S^2\dot{S}S + S^3\dot{S})\right\}$$

$$= -i\dot{S} + \underline{S\dot{S} - \frac{1}{2}(\dot{S}S + S\dot{S})} + \frac{i}{6}(\dot{S}S^2 + S\dot{S}S + S^2\dot{S}) - \frac{i}{2}(S\dot{S}S + S^2\dot{S}) + \frac{i}{2}S^2\dot{S} + O(S^4)$$

$$= -i\dot{S} + \frac{1}{2}(S\dot{S} - \dot{S}S) + i\left(\frac{1}{6}\dot{S}S^2 - \frac{1}{3}S\dot{S}S + \frac{1}{6}S^2\dot{S}\right)$$

$$= -i\dot{S} + \frac{1}{2}[S,\dot{S}] + \frac{i}{6}\left\{(\dot{S}S - S\dot{S})S - S(\dot{S}S - S\dot{S})\right\}$$

$$= -i\dot{S} + \frac{1}{2}[S,\dot{S}] + \frac{i}{6}\left([\dot{S},S]S - S[\dot{S},S]\right)$$

$$= -i\dot{S} + \frac{1}{2}[S,\dot{S}] + \frac{i}{6}[S,[S,\dot{S}]]$$

よって $$\boxed{\begin{aligned} H_D' = {} & H_D + i[S,H_D] - \frac{1}{2}[S,[S,H_D]] - \frac{i}{6}[S,[S,[S,H_D]]] + \frac{1}{24}[S,[S,[S,[S,H_D]]]] \\ & - \hbar\dot{S} - \frac{i}{2}\hbar[S,\dot{S}] + \frac{\hbar}{6}[S,[S,\dot{S}]] \end{aligned}}$$

$$[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B,C] + [A,C]B$$

$$H_D = c\alpha\cdot\left(\mathbb{P} - \frac{e}{c}A\right) + mc^2\beta + e\phi = c\mathcal{O} + mc^2\beta + \mathcal{E}$$

where $\mathcal{O} = \alpha\cdot(\mathbb{P} - \frac{e}{c}A)$, $\mathcal{E} = e\phi$

$$\mathcal{O}\beta = -\beta\mathcal{O}, \quad \mathcal{E}\beta = \beta\mathcal{E} \quad \rightarrow \quad [\mathcal{O},\beta] = -2\beta\mathcal{O}, \quad [\mathcal{E},\beta] = 0$$

free-particleの場合と同様にして、$S = -i\beta\alpha\cdot(\mathbb{P} - \frac{e}{c}A)/2mc = -i\frac{\beta\mathcal{O}}{2mc}$ とおく。

$$i[S, H_D] = i\left\{\left[-i\frac{\beta\mathcal{O}}{2mc},\ c\mathcal{O} + mc^2\beta + \mathcal{E}\right]\right\}$$

$$= \frac{1}{2mc}\left\{\beta\left[\mathcal{O}, c\mathcal{O} + mc^2\beta + \mathcal{E}\right] + \left[\beta, c\mathcal{O} + mc^2\beta + \mathcal{E}\right]\mathcal{O}\right\}$$

$$= \frac{1}{2mc}\cdot\left\{-2mc^2\mathcal{O} + 2c\beta\mathcal{O}^2 + \beta[\mathcal{O},\mathcal{E}]\right\}$$

$$= -c\mathcal{O} + \frac{\beta}{2mc}[\mathcal{O},\mathcal{E}] + \frac{1}{m}\beta\mathcal{O}^2$$

$$-\frac{1}{2}[S,[S,H_D]] = \frac{i}{2}\left[-i\frac{\beta\mathcal{O}}{2mc},\ -c\mathcal{O} + \frac{\beta}{2mc}[\mathcal{O},\mathcal{E}] + \frac{1}{m}\beta\mathcal{O}^2\right]$$

$$= \frac{1}{4mc}\left\{\beta\left[\mathcal{O}, -c\mathcal{O} + \frac{\beta}{2mc}[\mathcal{O},\mathcal{E}] + \frac{1}{m}\beta\mathcal{O}^2\right] + \left[\beta, -c\mathcal{O} + \frac{\beta}{2mc}[\mathcal{O},\mathcal{E}] + \frac{1}{m}\beta\mathcal{O}^2\right]\mathcal{O}\right\}$$

$$= \frac{1}{4mc}\left\{\frac{\beta}{2mc}\left(\mathcal{O}\beta[\mathcal{O},\mathcal{E}] - \beta[\mathcal{O},\mathcal{E}]\mathcal{O}\right) + \frac{\beta}{m}\left(\mathcal{O}\beta\mathcal{O}^2 - \beta\mathcal{O}^3\right)\right.$$

$$\left. - 2c\beta\mathcal{O}^2 + \frac{1}{2mc}\left([\mathcal{O},\mathcal{E}] - \beta[\mathcal{O},\mathcal{E}]\beta\right)\mathcal{O} + \frac{1}{m}\left(\mathcal{O}^2 - \beta\mathcal{O}^2\beta\right)\mathcal{O}\right\}$$

$$= \frac{1}{4mc}\left\{-\frac{1}{2mc}\left(\mathcal{O}[\mathcal{O},\mathcal{E}] + [\mathcal{O},\mathcal{E}]\mathcal{O}\right) - \frac{2}{m}\mathcal{O}^3 - 2c\beta\mathcal{O}^2\right.$$

$$\left. + \frac{1}{mc}[\mathcal{O},\mathcal{E}]\mathcal{O} + 0\right\}$$

$$= \frac{1}{4mc}\left\{\frac{1}{2mc}\left([\mathcal{O},\mathcal{E}]\mathcal{O} - \mathcal{O}[\mathcal{O},\mathcal{E}]\right) - 2c\beta\mathcal{O}^2 - \frac{2}{m}\mathcal{O}^3\right\}$$

$$= -\frac{\beta\mathcal{O}^2}{2m} - \frac{1}{8m^2c^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{1}{2m^2c}\mathcal{O}^3$$

$$-\frac{i}{6}[S,[S,[S,H_D]]] = \frac{i}{3}\left[S, -\frac{1}{2}[S,[S,H_D]]\right]$$

$$= \frac{i}{3}\left[S, -\frac{\beta\mathcal{O}^2}{2m} - \frac{1}{8m^2c^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{1}{2m^2c}\mathcal{O}^3\right]$$

$$= \frac{i}{3}\cdot\frac{-i}{2mc}\left[\beta\mathcal{O}, -\frac{\beta\mathcal{O}^2}{2m} - \frac{1}{8m^2c^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{1}{2m^2c}\mathcal{O}^3\right]$$

$$= \frac{1}{6mc}\Big\{-\frac{1}{2m}(\beta\mathcal{O}\beta\mathcal{O}^2-\beta\mathcal{O}^2\beta\mathcal{O}) - \frac{1}{8m^2c^2}\big(\beta\mathcal{O}[\mathcal{O},[\mathcal{O},\mathcal{E}]]-[\mathcal{O},[\mathcal{O},\mathcal{E}]]\beta\mathcal{O}\big)$$

$$-\frac{1}{2m^2c}(\beta\mathcal{O}^4-\mathcal{O}^3\beta\mathcal{O})\Big\}$$

$$= \frac{1}{6mc}\left\{\frac{\mathcal{O}^3}{m} - \frac{\beta}{8m^2c^2}[\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{E}]]] - \frac{\beta\mathcal{O}^4}{m^2c}\right\}$$

$$= \frac{1}{6mc}\left\{\frac{\mathcal{O}^3}{m} - \frac{\beta}{8m^2c^2}[\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{E}]]] - \frac{\beta\mathcal{O}^4}{m^2c}\right\}$$

$$= \frac{\mathcal{O}^3}{6m^2c} - \frac{\beta\mathcal{O}^4}{6m^3c^2} - \frac{\beta}{48m^3c^3}[\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{E}]]]$$

- $$\frac{1}{24}[S,[S,[S,[S,H_D]]]] = \frac{i}{4}\left[S, -\frac{i}{6}[S,[S,[S,H_D]]]\right]$$

$$= \frac{i}{4}\left[-\frac{i}{2mc}\beta\mathcal{O}, \frac{\mathcal{O}^3}{6m^2c} - \frac{\beta\mathcal{O}^4}{6m^3c^2} - \frac{\beta}{48m^3c^3}[\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{E}]]]\right]$$

$$= \frac{1}{8mc}\Big\{\frac{1}{6m^2c}(\beta\mathcal{O}^4-\mathcal{O}^3\beta\mathcal{O}) - \frac{1}{6m^3c^2}(\beta\mathcal{O}\beta\mathcal{O}^4-\beta\mathcal{O}^4\beta\mathcal{O})$$

$$-\frac{1}{48m^3c^3}\big(\beta\mathcal{O}\beta[\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{E}]]]-\beta[\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{E}]]]\beta\mathcal{O}\big)\Big\}$$

$$= \frac{1}{8mc}\left\{\frac{\beta\mathcal{O}^4}{3m^2c} + \frac{\beta\mathcal{O}^5}{3m^3c^2} + \frac{1}{48m^3c^3}[\mathcal{O},[\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{E}]]]]\right\}$$

$$= \frac{\beta\mathcal{O}^4}{24m^3c^2} + \frac{\beta\mathcal{O}^5}{24m^4c^3} + \frac{1}{384m^4c^3}[\mathcal{O},[\mathcal{O},[\mathcal{O},[\mathcal{O},\mathcal{E}]]]]$$

また.

$$-\hbar\dot{S} = i\hbar\frac{\beta\dot{O}}{2mc}$$

$$-\frac{i}{2}\hbar[S,\dot{S}] = -\frac{i\hbar}{2}\left[-i\frac{\beta O}{2mc},\ i\frac{\beta\dot{O}}{2mc}\right]$$

$$= -\frac{i\hbar}{2}\cdot\frac{1}{4m^2c^2}\cdot[\beta O,\ \beta\dot{O}]$$

$$= -\frac{i\hbar}{8m^2c^2}\left(\beta[O,\beta\dot{O}] + [\beta,\beta\dot{O}]O\right)$$

$$= +\frac{i\hbar}{8m^2c^2}\left(\beta O\beta\dot{O} - \beta\dot{O}\beta O\right) = -\frac{i\hbar}{8m^2c^2}[O,\dot{O}]$$

$$\frac{\hbar}{6}[S,[S,\dot{S}]] = \frac{i}{3}\left[S,\ -\frac{i}{2}\hbar[S,\dot{S}]\right]$$

$$= \frac{i}{3}\cdot\left[-i\frac{\beta O}{2mc},\ -\frac{i\hbar}{8m^2c^2}[O,\dot{O}]\right]$$

$$= \frac{-i\hbar}{48m^3c^3}\left(\beta O[O,\dot{O}] - [O,\dot{O}]\beta O\right)$$

$$= -\frac{i\hbar\beta}{48m^3c^3}[O,[O,\dot{O}]]$$

p^4 のオーダーで、$O(1/c^2)$ まで有効と考えると、

$$H_D' = -c\mathcal{O} + \frac{\beta}{2mc}[\mathcal{O},\mathcal{E}] + \frac{1}{m}\beta\mathcal{O}^2 - \frac{\beta\mathcal{O}^2}{2m} - \frac{1}{8m^2c^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{1}{2m^2c}\mathcal{O}^3$$

$$+ \frac{\mathcal{O}^3}{6m^2c} - \frac{\beta\mathcal{O}^4}{6m^3c^2} + \frac{\beta\mathcal{O}^4}{24m^3c^2} + i\hbar\frac{\beta\dot{\mathcal{O}}}{2mc} - \frac{i\hbar}{8m^2c^2}[\mathcal{O},\dot{\mathcal{O}}] + c\mathcal{O} + mc^2\beta + \mathcal{E}$$

$$= \beta\left(mc^2 + \frac{\mathcal{O}^2}{2m} - \frac{\mathcal{O}^4}{8m^3c^2}\right) + \mathcal{E} - \frac{1}{8m^2c^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{i\hbar}{8m^2c^2}[\mathcal{O},\dot{\mathcal{O}}]$$

$$\underbrace{+ \frac{\beta}{2mc}[\mathcal{O},\mathcal{E}] - \frac{\mathcal{O}^3}{3m^2c} + \frac{i\hbar}{2mc}\beta\dot{\mathcal{O}}}_{\text{mixing large and small component}}$$

$$= \beta mc^2 + \mathcal{E}' + c\mathcal{O}'$$

where,
$$\mathcal{E}' = \beta\left(\frac{\mathcal{O}^2}{2m} - \frac{\mathcal{O}^4}{8m^3c^2}\right) + \mathcal{E} - \frac{1}{8m^2c^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{i\hbar}{8m^2c^2}[\mathcal{O},\dot{\mathcal{O}}]$$

$$\mathcal{O}' = \frac{\beta}{2mc^2}[\mathcal{O},\mathcal{E}] - \frac{\mathcal{O}^3}{3m^2c^2} + \frac{i\hbar}{2mc^2}\beta\dot{\mathcal{O}}$$

ここで $S' = -i\dfrac{\beta\mathcal{O}'}{2mc} = \dfrac{-i\beta}{2mc}\left(\dfrac{\beta}{2mc^2}[\mathcal{O},\mathcal{E}] - \dfrac{\mathcal{O}^3}{3m^2c^2} + \dfrac{i\hbar\beta\dot{\mathcal{O}}}{2mc^2}\right)$ を用いて、 $S' = O\left(\frac{1}{c^3}\right)$

もう一度変換すると、

$$H_D'' = e^{iS'}\left(H_D' - i\hbar\frac{\partial}{\partial t}\right)e^{-iS'}$$

$$= H_D' + i[S',H_D'] - \frac{1}{2}[S',[S',H_D']] - \frac{i}{6}\underbrace{[S',[S',[S',H_D']]]}_{O\left(\frac{1}{c^4}\right)} - \hbar\dot{S}' - \frac{i\hbar}{2}[S',\dot{S}']$$

$$i[S',H_D'] = i\left[-i\frac{\beta\mathcal{O}'}{2mc},\ \beta mc^2 + \mathcal{E}' + \mathcal{O}'\right]$$

$$= \frac{1}{2mc}\left\{(\beta\mathcal{O}'\beta - \mathcal{O}')mc^2 + (\beta\mathcal{O}'\mathcal{E}' - \mathcal{E}'\beta\mathcal{O}')\right\} = \frac{1}{2mc}\left\{-2mc^2\mathcal{O}' + \beta[\mathcal{O}',\mathcal{E}']\right\}$$

$$= -c\mathcal{O}' + \frac{\beta}{2mc}[\mathcal{O}',\mathcal{E}']$$

$$-\frac{1}{2}[S',[S',H_D']] = \frac{i}{2}\left[S',\ -c\mathcal{O}' + \frac{\beta}{2mc}[\mathcal{O}',\mathcal{E}']\right]$$

$$= \frac{i}{2}\left(-i\frac{1}{2mc}\right)\left[\beta\mathcal{O}',\ -c\mathcal{O}' + \frac{\beta}{2mc}[\mathcal{O}',\mathcal{E}']\right] = \frac{1}{4mc}\left(-2c\beta\mathcal{O}'^2 - \frac{1}{2mc}[\mathcal{O}',[\mathcal{O}',\mathcal{E}']]\right)$$

$$\fallingdotseq -\frac{\beta\mathcal{O}'^2}{2m} \qquad \left(\because\ [\mathcal{O}',[\mathcal{O}',\mathcal{E}']] = O\left(\tfrac{1}{c^2}\right)\right)$$

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$$-\hbar\dot{S}' = -\hbar\cdot\left(-i\frac{\beta\dot{\mathcal{O}}'}{2mc}\right) = i\hbar\frac{\beta\dot{\mathcal{O}}'}{2mc}$$

よって、

$$H_D'' = \beta mc^2 + \mathcal{E}' + c\mathcal{O}' - c\mathcal{O}' + \frac{\beta}{2mc}[\mathcal{O}', \mathcal{E}'] + i\hbar\frac{\beta\dot{\mathcal{O}}'}{2mc}$$

$$= \beta mc^2 + \mathcal{E}' + \frac{\beta}{2mc}[\mathcal{O}', \mathcal{E}'] + i\hbar\frac{\beta\dot{\mathcal{O}}'}{2mc} = \beta mc^2 + \mathcal{E}' + c\mathcal{O}''$$ ($\mathcal{O}''$ は $1/c^4$ のオーダー)

$$S'' = -i\frac{\beta\mathcal{O}''}{2mc} = -i\frac{\beta}{2mc}\left(\frac{\beta}{2mc^2}[\mathcal{O}', \mathcal{E}'] + i\hbar\frac{\beta\dot{\mathcal{O}}'}{2mc^2}\right)$$ として、3回目の変換をすると、

$$H_D''' = H_D'' + i[S'', H_D''] = \beta mc^2 + \mathcal{E}' + c\mathcal{O}'' - c\mathcal{O}'' + \frac{\beta}{2mc}[\mathcal{O}'', \mathcal{E}']$$

$$= \beta mc^2 + \mathcal{E}'$$

$$= \beta\left(mc^2 + \frac{\mathcal{O}^2}{2m} - \frac{\mathcal{O}^4}{8m^3c^2}\right) + \mathcal{E} - \frac{1}{8m^2c^2}[\mathcal{O}, [\mathcal{O}, \mathcal{E}]] - \frac{i\hbar}{8m^2c^2}[\mathcal{O}, \dot{\mathcal{O}}]$$

ここで、$(\sigma\cdot A)(\sigma\cdot B) = \sigma^i A^i \sigma^j B^j = \frac{1}{2}\{\sigma^i, \sigma^j\}A^i B^j + \frac{1}{2}[\sigma^i, \sigma^j]A^i B^j$

$$= \delta^{ij}A^i B^j + i\varepsilon^{ijk}\sigma^k A^i B^j$$

$$= A\cdot B + i\sigma\cdot(A\times B)$$ より

$$P\times A = \varepsilon^{ijk}p^j A^k + \varepsilon^{ijk}A^k p^j$$
$$= (P\times A) - \varepsilon^{ijk}A^j p^k$$
$$= (P\times A) - A\times P$$

$$\left(\sigma\cdot\left(P - \frac{e}{c}A\right)\right)^2 = \left(P - \frac{e}{c}A\right)^2 + i\sigma\cdot\left\{\left(P - \frac{e}{c}A\right)\times\left(P - \frac{e}{c}A\right)\right\}$$

$$= \left(P - \frac{e}{c}A\right)^2 + i\sigma\cdot\left\{-\frac{e}{c}P\times A - \frac{e}{c}A\times P\right\}$$

$$= \left(P - \frac{e}{c}A\right)^2 + i\sigma\cdot\left\{-\frac{e}{c}(P\times A) + \frac{e}{c}A\times P - \frac{e}{c}A\times P\right\}$$

$$= \left(P - \frac{e}{c}A\right)^2 - i\frac{e}{c}\sigma\cdot(-i\hbar\nabla\times A)$$

$$= \left(P - \frac{e}{c}A\right)^2 - \frac{e\hbar}{c}\sigma\cdot B$$

よって、
$$\frac{\mathcal{O}^2}{2m} = \frac{\left(P - \frac{e}{c}A\right)^2}{2m} - \frac{e\hbar}{2mc}\sigma\cdot B$$

$$-\frac{1}{8m^2c^2}\left[\mathcal{O},[\mathcal{O},\mathcal{E}]\right]-\frac{i\hbar}{8m^2c^2}[\mathcal{O},\dot{\mathcal{O}}]$$

$$=-\frac{1}{8m^2c^2}\left[\mathcal{O},\left([\mathcal{O},\mathcal{E}]+i\hbar\dot{\mathcal{O}}\right)\right] \quad //$$

ここで、$\frac{1}{8m^2c^2}\left([\mathcal{O},\mathcal{E}]+i\hbar\dot{\mathcal{O}}\right)=\frac{1}{8m^2c^2}\left(\left[\alpha\cdot(\mathbb{P}-\frac{e}{c}\mathbb{A}),e\phi\right]+i\hbar\alpha\cdot(\dot{\mathbb{P}}-\frac{e}{c}\dot{\mathbb{A}})\right)$

$$=\frac{1}{8m^2c^2}\left([\alpha\cdot\mathbb{P},e\phi]+i\hbar\alpha\cdot\left(-\frac{e}{c}\dot{\mathbb{A}}\right)\right)$$

$$=\frac{e}{8m^2c^2}\left(-i\hbar(\alpha\cdot\nabla)\phi-\cancel{i\hbar\phi\alpha\cdot\nabla}+\cancel{i\hbar\phi\alpha\cdot\nabla}-i\frac{\hbar}{c}\alpha\cdot\dot{\mathbb{A}}\right)$$

$$=\frac{+i\hbar e}{8m^2c^2}\left(-\alpha\cdot\frac{\partial\mathbb{A}}{c\partial t}-\alpha\cdot(\nabla\phi)\right)=\frac{i\hbar e}{8m^2c^2}\alpha\cdot\mathbb{E}. \quad \text{より}$$

$$-\frac{1}{8m^2c^2}\left[\mathcal{O},\left([\mathcal{O},\mathcal{E}]+i\hbar\dot{\mathcal{O}}\right)\right]=-\left[\mathcal{O},\frac{i\hbar e}{8m^2c^2}\alpha\cdot\mathbb{E}\right]$$

$$=\frac{-i\hbar e}{8m^2c^2}\left[\alpha\cdot(\mathbb{P}-\frac{e}{c}\mathbb{A}),\alpha\cdot\mathbb{E}\right]=-\frac{i\hbar e}{8m^2c^2}\left[\alpha\cdot\mathbb{P};\alpha\cdot\mathbb{E}\right]$$

$$=-\frac{\hbar^2 e}{8m^2c^2}\left((\alpha\cdot\nabla)(\alpha\cdot\mathbb{E})-(\alpha\cdot\mathbb{E})(\alpha\cdot\mathbb{P})\right)$$

$$=-\frac{\hbar^2 e}{8m^2c^2}\left(\alpha^i\alpha^j(\nabla^iE^j)+\alpha^i\alpha^jE^j\nabla^i-\alpha^j\alpha^iE^j\nabla^i\right)$$

$$=-\frac{\hbar^2 e}{8m^2c^2}\left(\frac{1}{2}\{\alpha^i,\alpha^j\}(\nabla^iE^j)+\frac{1}{2}[\alpha^i,\alpha^j](\nabla^iE^j)+[\alpha^i,\alpha^j]E^j\nabla^i\right)$$

$$=-\frac{\hbar^2 e}{8m^2c^2}\left(\delta^{ij}(\nabla^iE^j)+i\varepsilon^{ijk}\sigma^k(\nabla^iE^j)+2i\varepsilon^{ijk}\sigma^kE^j\nabla^i\right)$$

$$=-\frac{\hbar^2 e}{8m^2c^2}\left(\nabla\cdot\mathbb{E}+i\sigma\cdot(\nabla\times\mathbb{E})+2i\sigma\cdot(\mathbb{E}\times\nabla)\right)$$

$$=-\frac{\hbar^2 e}{8m^2c^2}\mathrm{div}\,\mathbb{E}-\frac{i\hbar^2 e}{8m^2c^2}\sigma\cdot\mathrm{rot}\,\mathbb{E}-\frac{\hbar e}{4m^2c^2}\sigma\cdot(\mathbb{E}\times\mathbb{P})$$

NO.

DATE

$$\frac{\sigma^4}{8m^3c^2} = \frac{1}{2mc^2}\left(\frac{\sigma^2}{2m}\right)^2 = \frac{1}{2mc^2}\left\{\frac{(\mathbb{P}-\frac{e}{c}\mathbb{A})^2}{2m} - \frac{\hbar e}{2mc}\,\sigma\cdot\mathbb{B}\right\}^2$$

$$= \frac{1}{2mc^2}\left\{\frac{(\mathbb{P}-\frac{e}{c}\mathbb{A})^4}{4m^2} - \frac{\hbar e}{4m^2c}\left[(\mathbb{P}-\frac{e}{c}\mathbb{A})^2(\sigma\cdot\mathbb{B}) + (\sigma\cdot\mathbb{B})(\mathbb{P}-\frac{e}{c}\mathbb{A})^2\right] + \frac{\hbar^2e^2}{4m^2c^2}(\sigma\cdot\mathbb{B})^2\right\}$$

$$= \frac{(\mathbb{P}-\frac{e}{c}\mathbb{A})^4}{8m^3c^2} - \frac{\hbar e}{8m^3c^2}\left[(\mathbb{P}-\frac{e}{c}\mathbb{A})^2(\sigma\cdot\mathbb{B}) + (\sigma\cdot\mathbb{B})(\mathbb{P}-\frac{e}{c}\mathbb{A})^2\right] + \frac{\hbar^2e^2}{8m^3c^4}(\sigma\cdot\mathbb{B})^2$$

以上より、

$$H_D'' = \beta\left(mc^2 + \frac{(\mathbb{P}-\frac{e}{c}\mathbb{A})^2}{2m} - \frac{e\hbar}{2mc}\,\sigma\cdot\mathbb{B} - \frac{(\mathbb{P}-\frac{e}{c}\mathbb{A})^4}{8m^3c^2} + \frac{\hbar e}{8m^3c^3}\left[(\mathbb{P}-\frac{e}{c}\mathbb{A})^2(\sigma\cdot\mathbb{B}) + (\sigma\cdot\mathbb{B})(\mathbb{P}-\frac{e}{c}\mathbb{A})^2\right] - \frac{\hbar^2e^2}{8m^3c^4}(\sigma\cdot\mathbb{B})^2 \right)$$

$$+ e\phi - \frac{\hbar^2 e}{8m^2c^2}\,\mathrm{div}\,\mathbb{E} - \frac{i\hbar^2 e}{8m^2c^2}\,\sigma\cdot(\mathrm{rot}\,\mathbb{E}) - \frac{\hbar e}{4m^2c^2}\,\sigma\cdot(\mathbb{E}\times\mathbb{P})$$

$$= \beta\left(mc^2 + \frac{(\mathbb{P}-\frac{e}{c}\mathbb{A})^2}{2m} - \frac{(\mathbb{P}-\frac{e}{c}\mathbb{A})^4}{8m^3c^2} \right) + e\phi - \frac{e\hbar}{2mc}\beta(\sigma\cdot\mathbb{B}) - \frac{e^2\hbar^2}{8m^3c^4}\beta(\sigma\cdot\mathbb{B})^2$$

$$- \frac{i\hbar^2 e}{8m^2c^2}\,\sigma\cdot(\mathrm{rot}\,\mathbb{E}) - \frac{\hbar e}{4m^2c^2}\,\sigma\cdot(\mathbb{E}\times\mathbb{P}) - \frac{\hbar^2 e}{8m^2c^2}\,\mathrm{div}\,\mathbb{E}$$

$$+ \frac{\hbar e}{8m^3c^3}\beta\left[(\mathbb{P}-\frac{e}{c}\mathbb{A})^2(\sigma\cdot\mathbb{B}) + (\sigma\cdot\mathbb{B})(\mathbb{P}-\frac{e}{c}\mathbb{A})^2\right]$$

The terms in the first bracket give the expansion of $\sqrt{(\mathbb{P}-\frac{e}{c}\mathbb{A})^2 + m^2c^4}$