Compton散乱の全断面積

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2} \frac{1}{[m+\omega_K(1-\cos\theta)]^2}\left[\frac{\omega_K^2(1-\cos\theta)^2}{m\{m+\omega_K(1-\cos\theta)\}} + 1 + \cos^2\theta\right]$$

$$= \frac{\alpha^2}{2} \frac{1}{m^2\left[1+\frac{\omega_K}{m}(1-\cos\theta)\right]^2}\left[\frac{\omega_K^2(1-\cos\theta)^2}{m^2\left\{1+\frac{\omega_K}{m}(1-\cos\theta)\right\}} + 1 + \cos^2\theta\right]$$

$$= \frac{\alpha^2}{2m^2} \cdot \frac{1}{[1+a(1-\cos\theta)]^2}\left[\frac{a^2(1-\cos\theta)^2}{1+a(1-\cos\theta)} + 1 + \cos^2\theta\right]$$

$$\left(a = \frac{\omega_K}{m}\right)$$

この立体角について積分すると．

$$\sigma = \int_0^{2\pi} d\varphi \int_{-1}^{1} dz \, \frac{\alpha^2}{2m^2} \frac{1}{[1+a(1-z)]^2}\left[\frac{a^2(1-z)^2}{1+a(1-z)} + 1 + z^2\right]$$

$$\boxed{\frac{\alpha}{m} = r_0}$$

$$= \pi r_0^2 \int_{-1}^{1} dz \left[\frac{a^2(1-z)^2}{\{1+a(1-z)\}^3} + \frac{1+z^2}{\{1+a(1-z)\}^2}\right]$$

$$I_1 = \int_{-1}^{1} dz \, \frac{a^2(1-z)^2}{\{1+a(1-z)\}^3}, \quad I_2 = \int_{-1}^{1} dz \, \frac{1+z^2}{\{1+a(1-z)\}^2}$$

$$I_1 = \int_{-1}^{1} dz \, \frac{a^2(1-z)^2}{\{1+a(1-z)\}^3}$$

$t = 1+a(1-z)$ とおくと.

$dt = -a\,dz$,

$$\begin{array}{c|c} z & -1 \to 1 \\ \hline t & 1+2a \to 1 \end{array}$$

$$= -\frac{1}{a}\int_{1+2a}^{1} dt \, \frac{(t-1)^2}{t^3}$$

$$= \frac{1}{a}\int_{1}^{1+2a} dt \left(\frac{t^2-2t+1}{t^3} \right)$$

$$= \frac{1}{a}\int_{1}^{1+2a} dt \left(\frac{1}{t} - \frac{2}{t^2} + \frac{1}{t^3} \right)$$

$$= \frac{1}{a}\left[\log t + \frac{2}{t} - \frac{1}{2t^2} \right]_{1}^{1+2a}$$

$$= \frac{1}{a}\left[\log(1+2a) + 2\left(\frac{1}{1+2a} - 1 \right) - \frac{1}{2}\left(\frac{1}{(1+2a)^2} - 1 \right) \right]$$

$$= \frac{1}{a}\left[\log(1+2a) + 2 \cdot \frac{-2a}{1+2a} - \frac{1}{2} \cdot \frac{-4a-4a^2}{(1+2a)^2} \right]$$

$$= \frac{1}{a}\log(1+2a) - \frac{4}{1+2a} + \frac{2+2a}{(1+2a)^2}$$

$$= \frac{1}{a}\log(1+2a) + \frac{-4(1+2a)+2+2a}{(1+2a)^2}$$

$$= \frac{1}{a}\log(1+2a) + \frac{-2-6a}{(1+2a)^2}$$

$$= \frac{1}{a}\log(1+2a) - \frac{2(1+3a)}{(1+2a)^2}$$

$$I_2 = \int_{-1}^{1} dz \frac{1+z^2}{\{1+a(1-z)\}^2}$$

$t = 1+a(1-z)$ とおくと

$dt = -a\,dz$,

$$\begin{array}{c|c} z & -1 \to 1 \\ \hline t & 1+2a \to 1 \end{array}$$

$$\frac{t-1}{a} = 1-z \Leftrightarrow z = 1 - \frac{t-1}{a}$$

$$= \int_{1}^{1+2a} dt \cdot \frac{1}{a} \cdot \frac{1+\left(1-\frac{t-1}{a}\right)^2}{t^2}$$

$$= \frac{1}{a}\int_{1}^{1+2a} dt \, \frac{1}{t^2} \cdot \left\{1+1-\frac{2(t-1)}{a}+\frac{(t-1)^2}{a^2}\right\}$$

$$= \frac{1}{a^3}\int_{1}^{1+2a} dt \, \frac{1}{t^2}\left\{2a^2-2a(t-1)+(t^2-2t+1)\right\}$$

$$= \frac{1}{a^3}\int_{1}^{1+2a} dt \, \frac{1}{t^2}\left\{2a^2+2a+1-2(1+a)t+t^2\right\}$$

$$= \frac{1}{a^3}\int_{1}^{1+2a} dt \left\{\frac{2a^2+2a+1}{t^2}-\frac{2(1+a)}{t}+1\right\}$$

$$= \frac{1}{a^3}\left[-\frac{2a^2+2a+1}{t}-2(1+a)\log t+t\right]_{1}^{1+2a}$$

$$= \frac{1}{a^3}\left[-(2a^2+2a+1)\left(\frac{1}{1+2a}-1\right)-2(1+a)\log(1+2a)+2a\right]$$

$$= \frac{1}{a^3}\left[-(2a^2+2a+1)\frac{-2a}{1+2a} - 2(1+a)\log(1+2a) + 2a \right]$$

$$= \frac{1}{a^3}\left[\frac{2a(2a^2+2a+1)+2a(1+2a)}{1+2a} - 2(1+a)\log(1+2a) \right]$$

$$= \frac{1}{a^3}\left[\frac{2a(2a^2+4a+2)}{1+2a} - 2(1+a)\log(1+2a) \right]$$

$$= \frac{4(a+1)^2}{a^2(1+2a)} - \frac{2(1+a)}{a^3}\log(1+2a)$$

$$= \frac{4(1+a)^2}{a^2(1+2a)} - \frac{2(1+a)}{a^3}\log(1+2a)$$

したがって

$$\sigma = \pi r_0^2\left[\frac{4(1+a)^2}{a^2(1+2a)} - \frac{2(1+a)}{a^3}\log(1+2a) + \frac{1}{a}\log(1+2a) - \frac{2(1+3a)}{(1+2a)^2} \right]$$

$$= 2\pi r_0^2\left[\frac{1+a}{a^3}\left\{ \frac{2a(1+a)}{1+2a} - \log(1+2a) \right\} \right.$$

$$\left. + \frac{1}{2a}\log(1+2a) - \frac{1+3a}{(1+2a)^2} \right]$$

$$= \sigma_{Thom} \times \frac{3}{4}\left[\frac{1+a}{a^3}\left\{ \frac{2a(1+a)}{1+2a} - \log(1+2a) \right\} + \frac{1}{2a}\log(1+2a) - \frac{1+3a}{(1+2a)^2} \right]$$

ただし、$a = \frac{w_k}{m}$, $\sigma_{Thom} = \frac{8\pi r_0^2}{3}$

トムソンの全断面積