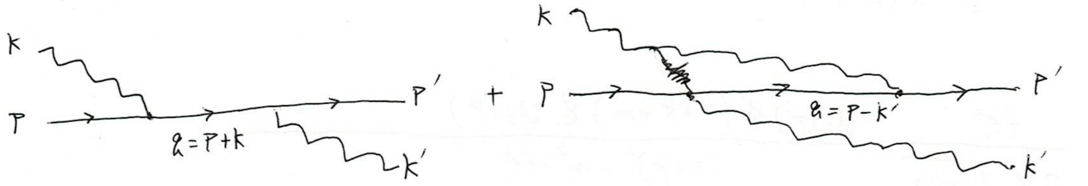
コンプトン散乱

過程



S_1　　　　　　　　　　　S_2

相互作用ハミルトニアン

$$H_{int} = -ie\int d^4x\, N\{\bar{\Psi}\not{A}\Psi\} \quad \text{より}$$

$$S_1 = 2\times\frac{(-ie)^2}{2!}\int d^4x_1\int d^4x_2 \langle p',s';k'|(\bar{\Psi}\not{A}\Psi)_{x_2}(\bar{\Psi}\not{A}\Psi)_{x_1}|p,s;k\rangle$$

$$= -e^2\int d^4x_1\int d^4x_2 \frac{1}{\sqrt{V}}\bar{u}_{s'}(p')e^{ip'\cdot x_2}\gamma^\mu\, iS_F(x_2-x_1)\gamma^\nu\frac{1}{\sqrt{V}}u_s(p)e^{-ip\cdot x_1}$$

$$\times\frac{1}{\sqrt{2\omega_{k'}V}}\varepsilon_\mu(k',\lambda')e^{ik'\cdot x_2}\frac{1}{\sqrt{2\omega_k V}}\varepsilon_\nu(k,\lambda)e^{-ik\cdot x_1}$$

$$= -\frac{ie^2}{2V^2\sqrt{\omega_k\omega_{k'}}}\int d^4x_1\int d^4x_2\,\bar{u}_{s'}(p')e^{ip'\cdot x_2}\gamma^\mu\int\frac{d^4q}{(2\pi)^4}\frac{\not{q}+m}{q^2-m^2+i\epsilon}e^{-iq\cdot(x_2-x_1)}$$

$$\times\gamma^\nu u_s(p)e^{-ip\cdot x_1}\varepsilon_\mu(k',\lambda')\varepsilon_\nu(k,\lambda)e^{ik'\cdot x_2}e^{-ik\cdot x_1}$$

$$= -\frac{ie^2}{2V^2\sqrt{\omega_k\omega_{k'}}}\int d^4x_1\int d^4x_2\int\frac{d^4q}{(2\pi)^4}\frac{\bar{u}_{s'}(p')\not{\varepsilon}(k',\lambda')(\not{q}+m)\not{\varepsilon}(k,\lambda)u_s(p)}{q^2-m^2+i\epsilon}$$

$$\times e^{i(q-p-k)\cdot x_1}\, e^{-i(q-p'-k')\cdot x_2}$$

$(2\pi)^4\delta^{(4)}(q-p-k)$　　　　$(2\pi)^4\delta^{(4)}(q-p'-k')$

$$= -\frac{ie^2}{2V^2\sqrt{\omega_k\omega_{k'}}}\int\frac{d^4q}{(2\pi)^4}\,\frac{\bar{u}_{s'}(p')\,\not{\varepsilon}(k',\lambda')\,(\not{q}+m)\,\not{\varepsilon}(k,\lambda)\,u_s(p)}{q^2-m^2+i\epsilon}$$

$$\times(2\pi)^4\,\delta^{(4)}(q-p-k)\;\times(2\pi)^4\,\delta^{(4)}(q-p'-k')$$

$$= -\frac{ie^2}{2V^2\sqrt{\omega_k\omega_{k'}}}\,\frac{\bar{u}_{s'}(p')\,\not{\varepsilon}'\,(\not{p}+\not{k}+m)\,\not{\varepsilon}\,u_s(p)}{(p+k)^2-m^2+i\epsilon}$$

$$\times(2\pi)^4\,\delta^{(4)}(p+k-p'-k')$$

$$S_2 = 2\times\frac{(-ie)^2}{2!}\int d^4x_1\int d^4x_2\,\langle p',s';k'|\,(\bar{\psi}\not{A}\psi)_{x_2}\,(\bar{\psi}\not{A}\psi)_{x_1}\,|p,s;k\rangle$$

$$= -e^2\int d^4x_1\int d^4x_2\,\frac{1}{\sqrt{V}}\,\bar{u}_{s'}(p')\,e^{ip'\cdot x_2}\,\gamma^\mu\, iS_F(x_2-x_1)\,\gamma^\nu\,\frac{1}{\sqrt{V}}\,u_s(p)\,e^{-ip\cdot x_1}$$

$$\times\frac{1}{\sqrt{2\omega_k V}}\,\varepsilon_\mu(k,\lambda)\,e^{-ik\cdot x_2}\quad\times\frac{1}{\sqrt{2\omega_{k'}V}}\,\varepsilon_\nu(k',\lambda')\,e^{ik'\cdot x_1}$$

$$= -\frac{ie^2}{2V^2\sqrt{\omega_k\omega_{k'}}}\int d^4x_1\int d^4x_2\,\bar{u}_s(p')\,e^{ip'\cdot x_2}\,\gamma^\mu\int\frac{d^4q}{(2\pi)^4}\,\frac{\not{q}+m}{q^2-m^2+i\epsilon}\,e^{-iq\cdot(x_2-x_1)}$$

$$\times\gamma^\nu\,u_s(p)\,e^{-ip\cdot x_1}\,\varepsilon_\mu(k,\lambda)\,\varepsilon_\nu(k',\lambda')\,e^{-ik\cdot x_2}\,e^{ik'\cdot x_1}$$

$$e^{-iq\cdot(-x_1)}\times e^{-ip\cdot x_1}\times e^{ik'\cdot x_1} = e^{+i(q-p+k')\cdot x_1}$$

$$e^{ip'\cdot x_2}\times e^{-iq\cdot x_2}\times e^{-ik\cdot x_2} = e^{-i(q-p'+k)\cdot x_2}$$

$$= -\frac{ie^2}{2V^2\sqrt{\omega_k \omega_{k'}}}\int d^4x_1 \int d^4x_2 \int \frac{d^4q}{(2\pi)^4} \frac{\bar{u}_{s'}(p')\not{\epsilon}(k,\lambda)(\not{q}+m)\not{\epsilon}(k',\lambda')u_s(p)}{q^2-m^2+i\epsilon}$$

$$\times e^{i(q-p+k')\cdot x_1} \times e^{-i(q-p'+k)\cdot x_2}$$

$$= -\frac{ie^2}{2V^2\sqrt{\omega_k \omega_{k'}}}\int \frac{d^4q}{(2\pi)^4} \frac{\bar{u}_{s'}(p')\not{\epsilon}(k,\lambda)(\not{q}+m)\not{\epsilon}(k',\lambda')u_s(p)}{q^2-m^2+i\epsilon}$$

$$\times (2\pi)^4\delta^{(4)}(q-p+k') \times (2\pi)^4\delta^{(4)}(q-p'+k)$$

$$q-p+k'-(q-p'+k)=0$$
$$\Leftrightarrow p-k'-p'+k=0$$
$$\Leftrightarrow p+k-p'-k'=0$$

$$= -\frac{ie^2}{2V^2\sqrt{\omega_k \omega_{k'}}} \frac{\bar{u}_{s'}(p')\not{\epsilon}(k,\lambda)(\not{p}-\not{k}'+m)\not{\epsilon}(k',\lambda')u_s(p)}{(p-k')^2-m^2+i\epsilon} \times (2\pi)^4\delta^{(4)}(p+k-p'-k')$$

よって

$$\boxed{\begin{aligned} S_1 &= -\frac{ie^2}{2V^2\sqrt{\omega_k \omega_{k'}}} \frac{\bar{u}_{s'}(p')\not{\epsilon}(k',\lambda')(\not{p}+\not{k}+m)\not{\epsilon}(k,\lambda)u_s(p)}{(p+k)^2-m^2+i\epsilon} \times (2\pi)^4\delta^{(4)}(p+k-p'-k') \\ S_2 &= -\frac{ie^2}{2V^2\sqrt{\omega_k \omega_{k'}}} \frac{\bar{u}_{s'}(p')\not{\epsilon}(k,\lambda)(\not{p}-\not{k}'+m)\not{\epsilon}(k',\lambda')u_s(p)}{(p-k')^2-m^2+i\epsilon} \times (2\pi)^4\delta^{(4)}(p+k-p'-k') \end{aligned}}$$

次に、電子の静止系に光子が入射する場合を考える。 ← 散乱断面積を計算する際、静止系で計算した方が簡単。

このとき、 $p = (m, 0, 0, 0)$ であるから、

エネルギー，運動量の保存則は

∘ $\mathbb{p}' = \mathbb{k} - \mathbb{k}'$

∘ $E' = \sqrt{(\mathbb{p}')^2 + m^2} = \sqrt{(\mathbb{k} - \mathbb{k}')^2 + m^2}$

$$= \sqrt{(\mathbb{k})^2 - 2\mathbb{k}\cdot\mathbb{k}' + (\mathbb{k}')^2 + m^2}$$

$$= \sqrt{m^2 + \omega_k^2 + \omega_{k'}^2 - 2\omega_k \omega_{k'} \cos\theta}$$

∘ $(p+k)^2 = p^2 + k^2 + 2p\cdot k = m^2 + 2p\cdot k$

∘ $(p-k')^2 = p^2 + (k')^2 - 2p\cdot k' = m^2 - 2p\cdot k'$

また、$(p+k)^2 = (p'+k')^2$ より、

$$m^2 + 2p\cdot k = m^2 + 2p'\cdot k' \Rightarrow \boxed{p\cdot k = p'\cdot k'}$$

$$\boxed{\begin{aligned} p\cdot k &= p^0 k_0 + p^i k_i \\ &= p^0 k^0 - \mathbb{p}\cdot\mathbb{k} \end{aligned}}$$

さらに、$k'\cdot(p+k) = k'\cdot(p'+k')$ より

$$p\cdot k' + k\cdot k' = \underbrace{p'\cdot k'}_{=p\cdot k} \Leftrightarrow p\cdot k = p\cdot k' + k\cdot k'$$

これより、$\omega_{k'}$ を ω_k で表せて、電子の静止系において

$$p\cdot k = p^0 k^0 - \underbrace{\mathbb{p}}_{0}\cdot\mathbb{k} = m\omega_k$$

$$p\cdot k' + k\cdot k' = p^0 \omega_{k'} - \underbrace{\mathbb{p}}_{0}\cdot\mathbb{k}' + \omega_k\omega_{k'} - \omega_k\omega_{k'}\cos\theta$$

$$= m\omega_{k'} + \omega_k\omega_{k'} - \omega_k\omega_{k'}\cos\theta$$

$$= \omega_{k'}(m + \omega_k(1-\cos\theta))$$

$$\therefore \boxed{\omega_{k'} = \frac{m\omega_k}{m + \omega_k(1-\cos\theta)}}$$

偏極については、

$\varepsilon^{\mu}=(0,\boldsymbol{\varepsilon}(\mathbb{k},\lambda))$, $\varepsilon'^{\mu}=(0,\boldsymbol{\varepsilon}(\mathbb{k}',\lambda'))$ を用いると、Lorentz条件から

$$\begin{cases} k\cdot\varepsilon=-\mathbb{k}\cdot\boldsymbol{\varepsilon}=0 \\ k'\cdot\varepsilon'=-\mathbb{k}'\cdot\boldsymbol{\varepsilon}'=0 \end{cases}$$

$\mathbb{P}=0$ により、$\boxed{\begin{gathered} P\cdot\varepsilon=0, \\ P\cdot\varepsilon'=0 \end{gathered}}$ も同時に成り立つ。

まとめると、

$$\mathbb{P}=\mathbb{k}-\mathbb{k}', \quad E'=\sqrt{m^2+\omega_k^2+\omega_{k'}^2-2\omega_k\omega_{k'}\cos\theta}$$

$$(P+k)^2 = m^2+2P\cdot k$$

$$(P-k')^2 = m^2-2P\cdot k'$$

$$P\cdot k=P'\cdot k'$$

$$k\cdot\varepsilon=-\mathbb{k}\cdot\boldsymbol{\varepsilon}=0$$

$$k'\cdot\varepsilon'=-\mathbb{k}'\cdot\boldsymbol{\varepsilon}'=0$$

$$P\cdot\varepsilon=0, \ \cancel{P'\cdot\boldsymbol{\varepsilon}=0} \quad (\mathbb{P}=0\text{のとき})$$

$$\omega_{k'}=\frac{m\omega_k}{m+\omega_k(1-\cos\theta)}$$

$P\cdot\varepsilon'=0$

この条件を使って、S_1とS_2を書き換えたい。

まず、S_1の $\bar{u}_{s'}(p')\not{\varepsilon}(k',\lambda')(\not{p}+\not{k}+m)\not{\varepsilon}(k,\lambda)u_s(p)$ に注目すると、

$$(\not{p}+m)\not{\varepsilon}(k,\lambda)u_s(p)$$

$$=\{2\underbrace{(p\cdot\varepsilon)}_{0}-\not{\varepsilon}\not{p}+\underset{\not{\varepsilon}}{m}\}u_s(p)$$

静止系の条件

$\not{a}\not{b}=2(a\cdot b)-\not{b}\not{a}$

Dirac方程式

$(\not{p}-m)u_s(p)=0$

$$=\not{\varepsilon}(-m+m)u_s(p)=0$$

により、

$$\boxed{\begin{aligned}&\bar{u}_{s'}(p')\not{\varepsilon}(k',\lambda')(\not{p}+\not{k}+m)\not{\varepsilon}(k,\lambda)u_s(p)\\&=\bar{u}_{s'}(p')\not{\varepsilon}(k',\lambda')\not{k}\not{\varepsilon}(k,\lambda)u_s(p)\end{aligned}}$$

同様に、S_2においても

$\cancel{\bar{u}_s(p')\not{\varepsilon}(k',\lambda')}$

$$\boxed{\begin{aligned}&\bar{u}_{s'}(p')\not{\varepsilon}(k,\lambda)(\not{p}-\not{k}'+m)\not{\varepsilon}(k',\lambda')u_s(p)\\&=-\bar{u}_{s'}(p')\not{\varepsilon}(k,\lambda)\not{k}'\not{\varepsilon}(k',\lambda')u_s(p)\end{aligned}}$$

また、

$$(p+k)^2-m^2=m^2+2p\cdot k-m^2=2p\cdot k$$

$$(p-k')^2-m^2=m^2-2p\cdot k'-m^2=-2p\cdot k'$$

S_1とS_2は、静止系において.

$$S_1 = -\frac{ie^2}{2V^2\sqrt{\omega_k \omega_{k'}}} \frac{\bar{u}_{s'}(p')\,\not{\epsilon}'\,\not{k}\,\not{\epsilon}\,u_s(p)}{2(p\cdot k)+i\epsilon} \times (2\pi)^4 \delta^{(4)}(p+k-p'-k')$$

$$S_2 = -\frac{ie^2}{2V^2\sqrt{\omega_k \omega_{k'}}} \frac{\bar{u}_{s'}(p')\,\not{\epsilon}\,\not{k}'\,\not{\epsilon}'\,u_s(p)}{2(p\cdot k')-i\epsilon} \times (2\pi)^4 \delta^{(4)}(p+k-p'-k')$$

よって散乱振幅は

$$\boxed{\begin{aligned} S &= S_1 + S_2 \\ &= -\frac{ie^2}{2V^2\sqrt{\omega_k \omega_{k'}}}\left[\frac{\bar{u}_{s'}(p')\,\not{\epsilon}'\,\not{k}\,\not{\epsilon}\,u_s(p)}{2(p\cdot k)+i\epsilon} + \frac{\bar{u}_{s'}(p')\,\not{\epsilon}\,\not{k}'\,\not{\epsilon}'\,u_s(p)}{2(p\cdot k')-i\epsilon}\right] \times (2\pi)^4 \delta^{(4)}(p+k-p'-k') \end{aligned}}$$

$$\begin{cases} T_1 = \bar{u}_{s'}(p')\,\not{\epsilon}'\,\not{k}\,\not{\epsilon}\,u_s(p) \\ T_2 = \bar{u}_{s'}(p')\,\not{\epsilon}\,\not{k}'\,\not{\epsilon}'\,u_s(p) \end{cases}$$

として. 絶対値の自乗を計算すると.

$$|S|^2 = \frac{e^4}{4V^4\omega_k\omega_{k'}}\left[\frac{|T_1|^2}{4(p\cdot k)^2} + \frac{|T_2|^2}{4(p\cdot k')^2} + \frac{T_1T_2^* + T_1^*T_2}{4(p\cdot k)(p\cdot k')}\right] \times (2\pi)^8 \left[\delta^{(4)}(p+k-p'-k')\right]^2$$

それぞれの始状態平均、終状態の和をとるため.

スピンの平均と和をとる

$$\circ\ \frac{1}{2}\sum_{s,s'}|T_1|^2 = \frac{1}{2}\sum_{s,s'}\left[\bar{u}_{s'}(p')\not\epsilon'\not k\not\epsilon u_s(p)\right]\left[\bar{u}_{s'}(p')\not\epsilon'\not k\not\epsilon u_s(p)\right]^*$$

$$= \frac{1}{2}\sum_{s,s'}\bar{u}_{s'}(p')_a\not\epsilon'_{ab}\not k_{bc}\not\epsilon_{cd}u_s(p)_d\,\underline{\bar{u}_{s'}(p')^*_e\not\epsilon'^*_{ef}\not k^*_{fg}\not\epsilon^*_{gh}u_s(p)^*_h}$$

ここで

$$\left[\bar{u}_{s'}(p')\not\epsilon'\not k\not\epsilon u_s(p)\right]^* = \bar{u}_{s'}(p')^*_e\not\epsilon'^*_{ef}\not k^*_{fg}\not\epsilon^*_{gh}u_s(p)^*_h$$ ← 転置をとる.

$$= u_s(p)^*_h\not\epsilon^\dagger_{hg}\not k^\dagger_{gf}\not\epsilon'^\dagger_{fe}\left(\gamma^0 u_{s'}(p')\right)_e$$

$$= u_s(p)^*_h\left(\gamma^0\not\epsilon\gamma^0\right)_{hg}\left(\gamma^0\not k\gamma^0\right)_{gf}\left(\gamma^0\not\epsilon'\gamma^0\right)_{fe}\left(\gamma^0 u_{s'}(p')\right)_e$$

$$= \underline{u_s(p)^*_h\left(\gamma^0\right.}\not\epsilon\not k\not\epsilon'\underbrace{\left.\gamma^0\right)_{he}\left(\gamma^0\right.}_{1}\left.u_{s'}(p')\right)_e$$

$$= \bar{u}_s(p)_h\left(\not\epsilon\not k\not\epsilon'\right)_{he}u_{s'}(p')_e$$

$$= \bar{u}_s(p)_e\not\epsilon_{ef}\not k_{fg}\not\epsilon'_{gh}u_{s'}(p')_h$$

よって.

$$\frac{1}{2}\sum_{s,s'}|T_1|^2 = \frac{1}{2}\sum_{s,s'}\bar{u}_{s'}(p')_a\not\epsilon'_{ab}\not k_{bc}\not\epsilon_{cd}u_s(p)_d\bar{u}_s(p)_e\not\epsilon_{ef}\not k_{fg}\not\epsilon'_{gh}u_{s'}(p')_h$$

$$= \frac{1}{2}\sum_{s,s'}\not\epsilon'_{ab}\not k_{bc}\not\epsilon_{cd}u_s(p)_d\bar{u}_s(p)_e\not\epsilon_{ef}\not k_{fg}\not\epsilon'_{gh}u_{s'}(p')_h\bar{u}_{s'}(p')_a$$

$$\sum_s u_s(p)_a\bar{u}_s(p)_b = \frac{(\not p+m)_{ab}}{2E_p}$$

$$= \frac{1}{2}\mathrm{Tr}\left[\not\epsilon'\not k\not\epsilon\frac{\not p+m}{2E_p}\not\epsilon\not k\not\epsilon'\frac{\not p'+m}{2E_{p'}}\right]$$

$$= \frac{1}{8E_pE_{p'}}\mathrm{Tr}\left[\not\epsilon'\not k\not\epsilon(\not p+m)\not\epsilon\not k\not\epsilon'(\not p'+m)\right]$$

$$= \frac{1}{8E_p E_{p'}} \left\{ \mathrm{Tr}\left[\underbrace{\not{\varepsilon}' \not{k} \not{\varepsilon} \not{p} \not{\varepsilon} \not{k} \not{\varepsilon}' \not{p}'}_{Ⓐ} \right] + m^2 \mathrm{Tr}\left[\underbrace{\not{\varepsilon}' \not{k} \not{\varepsilon} \not{\varepsilon} \not{k} \not{\varepsilon}'}_{Ⓑ} \right] \right\}$$

$$Ⓐ = \mathrm{Tr}\left[\not{\varepsilon}' \not{k} \not{\varepsilon} \not{p} \not{\varepsilon} \not{k} \not{\varepsilon}' \not{p}' \right]$$

$$= \mathrm{Tr}\left[\not{k} \not{\varepsilon} \not{p} \not{\varepsilon} \not{k} \not{\varepsilon}' \not{p}' \not{\varepsilon}' \right]$$

$$= \mathrm{Tr}\left[\not{k} \left\{ 2(\underbrace{p \cdot \varepsilon}_{0}) - \not{p} \not{\varepsilon} \right\} \not{\varepsilon} \not{k} \left\{ 2(p' \cdot \varepsilon') - \not{p}' \not{\varepsilon}' \right\} \not{\varepsilon}' \right]$$

$$= \mathrm{Tr}\left[-\not{k} \not{p} \underbrace{\not{\varepsilon} \not{\varepsilon}}_{-1} \not{k} \times 2(p' \cdot \varepsilon') \not{\varepsilon}' \right] + \mathrm{Tr}\left[\not{k} \not{p} \underbrace{\not{\varepsilon} \not{\varepsilon}}_{-1} \not{k} \not{p}' \underbrace{\not{\varepsilon}' \not{\varepsilon}'}_{-1} \right]$$

$$= 2(\underbrace{p' \cdot \varepsilon'}_{k \cdot \varepsilon'}) \mathrm{Tr}\left[\not{k} \not{p} \not{k} \not{\varepsilon}' \right] + \mathrm{Tr}\left[\not{k} \not{p} \not{k} \not{p}' \right]$$

チェック

$p + k = p' + k'$ より

$(p + k) \cdot \varepsilon' = (p' + k') \cdot \varepsilon'$

$\Leftrightarrow \underbrace{p \cdot \varepsilon'}_{0} + k \cdot \varepsilon' = p' \cdot \varepsilon' + \underbrace{k' \cdot \varepsilon'}_{0}$

$\therefore \ p' \cdot \varepsilon' = k \cdot \varepsilon'$

$$= 8(k \cdot \varepsilon') \left[(p \cdot k)(k \cdot \varepsilon') - \underbrace{k^2}_{0} (p \cdot \varepsilon') + (k \cdot \varepsilon')(p \cdot k) \right]$$

$$+ 4 \left[(p \cdot k)(\underbrace{p' \cdot k}_{p \cdot k'}) - \underbrace{k^2}_{0} (p \cdot p') + (\underbrace{p' \cdot k}_{p \cdot k'})(p \cdot k) \right]$$

$$= 16 (p \cdot k)(k \cdot \varepsilon')^2 + 8 (p \cdot k)(p \cdot k')$$

$$= 8 (p \cdot k) \left[2 (k \cdot \varepsilon')^2 + (p \cdot k') \right]$$

$$Ⓑ = m^2 \mathrm{Tr}\left[\not{\varepsilon}' \not{k} \underbrace{\not{\varepsilon} \not{\varepsilon}}_{-1} \not{k} \not{\varepsilon}' \right]$$

$$= -m^2 \mathrm{Tr}\left[\not{\varepsilon}' \underbrace{\not{k} \not{k}}_{k^2 = 0} \not{\varepsilon}' \right]$$

$$= 0$$

これにより、

$$\frac{1}{2}\sum_{s,s'}|T_1|^2 = \frac{1}{8E_pE_{p'}}\cdot 8(p\cdot k)\left[2(k\cdot\varepsilon')^2+(p\cdot k')\right]$$

$$= \frac{(p\cdot k)\left[2(k\cdot\varepsilon')^2+(p\cdot k')\right]}{E_pE_{p'}}$$

次

$$\circ\ \frac{1}{2}\sum_{s,s'}|T_2|^2 = \frac{1}{2}\sum_{s,s'}\left[\bar{u}_{s'}(p')\not{\varepsilon}\not{k}'\not{\varepsilon}'u_s(p)\right]\left[\bar{u}_{s'}(p')\not{\varepsilon}\not{k}'\not{\varepsilon}'u_s(p)\right]^*$$

$$= \frac{1}{2}\sum_{s,s'}\bar{u}_{s'}(p')_a\not{\varepsilon}_{ab}\not{k}'_{bc}\not{\varepsilon}'_{cd}u_s(p)_d\,\underbrace{\bar{u}_{s'}(p')^*_e\not{\varepsilon}^*_{ef}\not{k}'^*_{fg}\not{\varepsilon}'^*_{gh}u_s(p)^*_h}_{\text{転置をとる}}$$

$$= \frac{1}{2}\sum_{s,s'}\bar{u}_{s'}(p')_a\not{\varepsilon}_{ab}\not{k}'_{bc}\not{\varepsilon}'_{cd}u_s(p)_d\,u_s(p)^*_h\not{\varepsilon}'^\dagger_{hg}\not{k}'^\dagger_{gf}\not{\varepsilon}^\dagger_{fe}\bar{u}_{s'}(p')^*_e$$

$$= \frac{1}{2}\sum_{s,s'}\bar{u}_{s'}(p')_a\not{\varepsilon}_{ab}\not{k}'_{bc}\not{\varepsilon}'_{cd}u_s(p)_d\,u_s(p)^*_h(\gamma^0\not{\varepsilon}'\gamma^0)_{hg}(\gamma^0\not{k}'\gamma^0)_{gf}(\gamma^0\not{\varepsilon}\gamma^0)_{fe}\bar{u}_{s'}(p')^*_e$$

$$= \frac{1}{2}\sum_{s,s'}\bar{u}_{s'}(p')_a\not{\varepsilon}_{ab}\not{k}'_{bc}\not{\varepsilon}'_{cd}u_s(p)_d\,\bar{u}_s(p)_h\not{\varepsilon}'_{hg}\not{k}'_{gf}\not{\varepsilon}_{fe}u_{s'}(p')_e$$

$$= \frac{1}{2}\sum_{s,s'}\not{\varepsilon}_{ab}\not{k}'_{bc}\not{\varepsilon}'_{cd}u_s(p)_d\,\bar{u}_s(p)_h\not{\varepsilon}'_{hg}\not{k}'_{gf}\not{\varepsilon}_{fe}u_{s'}(p')_e\bar{u}_{s'}(p')_a$$

$$= \frac{1}{2}\mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{\varepsilon}'\frac{\not{p}+m}{2E_p}\not{\varepsilon}'\not{k}'\not{\varepsilon}\frac{\not{p}'+m}{2E_{p'}}\right]$$

$$= \frac{1}{8E_pE_{p'}}\mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{\varepsilon}'(\not{p}+m)\not{\varepsilon}'\not{k}'\not{\varepsilon}(\not{p}'+m)\right]$$

$$= \frac{1}{8E_pE_{p'}}\left\{\underbrace{\mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}'\not{k}'\not{\varepsilon}\not{p}'\right]}_{A} + m^2\underbrace{\mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{\varepsilon}'\not{k}'\not{\varepsilon}\right]}_{B}\right\}$$

$$Ⓐ = \mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}'\not{k}'\not{\varepsilon}\not{p}'\right]$$

$$= \mathrm{Tr}\left[\not{\varepsilon}\not{p}'\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}'\not{k}'\right]$$

$$= \mathrm{Tr}\left[\left\{2(p'\cdot\varepsilon) - \not{p}'\not{\varepsilon}\right\}\not{\varepsilon}\not{k}'\left\{2\underbrace{(p\cdot\varepsilon')}_{0} - \not{p}\not{\varepsilon}'\right\}\not{\varepsilon}'\not{k}'\right]$$

$$= -2\underbrace{(p'\cdot\varepsilon)}_{-k'\cdot\varepsilon}\mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{p}\underbrace{\not{\varepsilon}'\not{\varepsilon}'}_{-1}\not{k}'\right] + \mathrm{Tr}\left[\not{p}'\underbrace{\not{\varepsilon}\not{\varepsilon}}_{-1}\not{k}'\not{p}\underbrace{\not{\varepsilon}'\not{\varepsilon}'}_{-1}\not{k}'\right]$$

チェック

$p+k = p'+k'$ より

$(p+k)\cdot\varepsilon = (p'+k')\cdot\varepsilon$

$\Leftrightarrow \underbrace{p\cdot\varepsilon}_{0} + \underbrace{k\cdot\varepsilon}_{0} = p'\cdot\varepsilon + k'\cdot\varepsilon$

$\therefore\ p'\cdot\varepsilon = -k'\cdot\varepsilon$

$$= -2(k'\cdot\varepsilon)\,\mathrm{Tr}\left[\not{\varepsilon}\not{k}'\not{p}\not{k}'\right] + \mathrm{Tr}\left[\not{p}'\not{k}'\not{p}\not{k}'\right]$$

$= -$[redacted]$8(k'\cdot\varepsilon)\left[(k'\cdot\varepsilon)(p\cdot k') - (p\cdot\varepsilon)\underbrace{k'^2}_{0} + (k'\cdot\varepsilon)(p\cdot k')\right]$

$+\ 4$[redacted]$\left[\underbrace{(p'\cdot k')}_{p\cdot k}(p\cdot k') - (p'\cdot p)\underbrace{k'^2}_{0} + \underbrace{(p'\cdot k')}_{p\cdot k}(p\cdot k')\right]$

$$= -16(p\cdot k')(k'\cdot\varepsilon)^2 + 8(p\cdot k')(p\cdot k)$$

$$= -8(p\cdot k')\left[2(k'\cdot\varepsilon)^2 - (p\cdot k)\right]$$

$$Ⓑ = m^2\,\mathrm{Tr}\left[\not{\varepsilon}\not{k}'\underbrace{\not{\varepsilon}'\not{\varepsilon}'}_{-1}\not{k}'\not{\varepsilon}\right]$$

$$= -m^2\,\mathrm{Tr}\left[\not{\varepsilon}\underbrace{\not{k}'\not{k}'}_{k'^2=0}\not{\varepsilon}\right]$$

$$= 0$$

よって

$\dfrac{1}{2}\sum_{s,s'}|T_2|^2 = \dfrac{-\text{[redacted]}(p\cdot k')\left[2(k'\cdot\varepsilon)^2 - (p\cdot k)\right]}{E_p E_{p'}}$

$$\circ\ \frac{1}{2}\sum_{s,s'} T_1 T_2^* = \frac{1}{2}\sum_{s,s'} \bar{u}_{s'}(p')\,\not{\epsilon}'\not{k}\not{\epsilon}\,u_s(p)\left[\bar{u}_{s'}(p')\not{\epsilon}\not{k}'\not{\epsilon}'u_s(p)\right]^*$$

$$= \frac{1}{2}\sum_{s,s'} \bar{u}_{s'}(p')_a \not{\epsilon}'_{ab}\not{k}_{bc}\not{\epsilon}_{cd}\,u_s(p)_d\,\underline{\bar{u}_{s'}(p')^*_e \not{\epsilon}^*_{ef}\not{k}'^*_{fg}\not{\epsilon}'^*_{gh}u_s(p)^*_h}$$

転置をとる.

$$= \frac{1}{2}\sum_{s,s'} \bar{u}_{s'}(p')_a \not{\epsilon}'_{ab}\not{k}_{bc}\not{\epsilon}_{cd}\,u_s(p)_d\,u_s(p)^*_h \not{\epsilon}'^{\dagger}_{hg}\not{k}'^{\dagger}_{gf}\not{\epsilon}^{\dagger}_{fe}\bar{u}_{s'}(p')^*_e$$

$$= \frac{1}{2}\sum_{s,s'} \bar{u}_{s'}(p')_a \not{\epsilon}'_{ab}\not{k}_{bc}\not{\epsilon}_{cd}\,u_s(p)_d\,u_s(p)^{\dagger}_h(\gamma^0\not{\epsilon}'\gamma^0)_{hg}(\gamma^0\not{k}'\gamma^0)_{gf}(\gamma^0\not{\epsilon}\gamma^0)_{fe}(\gamma_0 u_{s'}(p'))_e$$

$$= \frac{1}{2}\sum_{s,s'} \bar{u}_{s'}(p')_a \not{\epsilon}'_{ab}\not{k}_{bc}\not{\epsilon}_{cd}\,u_s(p)_d\,\bar{u}_s(p)_h\not{\epsilon}'_{hg}\not{k}'_{gf}\not{\epsilon}_{fe}u_{s'}(p')_e$$

$$= \frac{1}{2}\sum_{s,s'} \not{\epsilon}'_{ab}\not{k}_{bc}\not{\epsilon}_{cd}\,u_s(p)_d\,\bar{u}_s(p)_h\not{\epsilon}'_{hg}\not{k}'_{gf}\not{\epsilon}_{fe}u_{s'}(p')_e\bar{u}_{s'}(p')_a$$

$$= \frac{1}{2}\mathrm{Tr}\left[\not{\epsilon}'\not{k}\not{\epsilon}\,\frac{\not{p}+m}{2E_p}\,\not{\epsilon}'\not{k}'\not{\epsilon}\,\frac{\not{p}'+m}{2E_{p'}}\right]$$

$$\boxed{\sum_s u_s(p)\bar{u}_s(p) = \frac{\not{p}+m}{2E_p}}$$

$$= \frac{1}{8E_pE_{p'}}\mathrm{Tr}\left[\not{\epsilon}'\not{k}\not{\epsilon}(\not{p}+m)\not{\epsilon}'\not{k}'\not{\epsilon}(\not{p}'+m)\right]$$

← 奇数個の γ-trace は消える。

$$= \frac{1}{8E_pE_{p'}}\Big\{\underbrace{\mathrm{Tr}\left[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\not{p}'\right]}_{Ⓐ} + \underbrace{m^2\,\mathrm{Tr}\left[\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}'\not{k}'\not{\epsilon}\right]}_{Ⓑ}\Big\}$$

$$Ⓐ = \mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\not{p}']$$

← エネルギー・運動量保存

$p+k = p'+k'$

$\Leftrightarrow$ $p' = p+k-k'$ を使う.

$$= \mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}(\not{p}+\not{k}-\not{k}')]$$

$$= \underbrace{\mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\not{p}]}_{①} + \underbrace{\mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\not{k}]}_{②} - \underbrace{\mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\not{k}']}_{③}$$

$$Ⓐ\text{-}① = \mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\not{p}]$$

$$= \mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\{2\underbrace{(p\cdot\epsilon')}_{0} - \not{\epsilon}'\not{p}\}\not{k}'\not{\epsilon}\not{p}]$$

$$= -\mathrm{Tr}[\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}'\not{p}\not{k}'\not{\epsilon}\not{p}]$$

$$= -\mathrm{Tr}[\{2(k\cdot\epsilon') - \not{k}\not{\epsilon}'\}\{2(\epsilon\cdot\epsilon') - \not{\epsilon}'\not{\epsilon}\}\not{p}\not{k}'\not{\epsilon}\not{p}]$$

$$= -4(k\cdot\epsilon')(\epsilon\cdot\epsilon')\mathrm{Tr}[\not{p}\not{k}'\not{\epsilon}\not{p}] + 2(k\cdot\epsilon')\mathrm{Tr}[\not{\epsilon}'\not{\epsilon}\not{p}\not{k}'\not{\epsilon}\not{p}]$$

$$+2(\epsilon\cdot\epsilon')\mathrm{Tr}[\not{k}\not{\epsilon}'\not{p}\not{k}'\not{\epsilon}\not{p}] - \mathrm{Tr}[\not{k}\underbrace{\not{\epsilon}'\not{\epsilon}'}_{-1}\not{\epsilon}\not{p}\not{k}'\not{\epsilon}\not{p}]$$

$$\boxed{\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}}$$

$$= -4(k\cdot\epsilon')(\epsilon\cdot\epsilon')\mathrm{Tr}[\not{k}'\not{\epsilon}\underbrace{\not{p}\not{p}}_{m^2}]$$

$$+2(k\cdot\epsilon')\mathrm{Tr}[\not{\epsilon}'\{2\underbrace{(p\cdot\epsilon)}_{0} - \not{p}\not{\epsilon}\}\{2(k'\cdot\epsilon) - \not{\epsilon}\not{k}'\}\not{p}]$$

$$+2(\epsilon\cdot\epsilon')\mathrm{Tr}[\not{k}\not{\epsilon}'\{2(p\cdot k') - \not{k}'\not{p}\}\{2\underbrace{(p\cdot\epsilon)}_{0} - \not{p}\not{\epsilon}\}]$$

$$+\mathrm{Tr}[\not{k}\not{\epsilon}\not{p}\not{k}'\not{\epsilon}\not{p}]$$

$$
\begin{aligned}
= & -16\, m^2 (\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) \\
& -4(k\cdot\varepsilon')(k'\cdot\varepsilon)\, \mathrm{Tr}\left[\not{\varepsilon}'\underbrace{\not{p}\not{\varepsilon}}\not{p}\right] + 2(k\cdot\varepsilon')\,\mathrm{Tr}\left[\not{\varepsilon}'\not{p}\underbrace{\not{\varepsilon}\not{\varepsilon}}_{-1}\not{k}'\not{p}\right] \\
& -4(\varepsilon\cdot\varepsilon')(p\cdot k')\,\mathrm{Tr}\left[\not{k}\not{\varepsilon}'\not{p}\not{\varepsilon}\right] + 2(\varepsilon\cdot\varepsilon')\,\mathrm{Tr}\left[\not{k}\not{\varepsilon}'\not{k}'\underbrace{\not{p}\not{p}}_{m^2}\not{\varepsilon}\right] \\
& + \mathrm{Tr}\left[\left\{2\underbrace{(k\cdot\varepsilon)}_{0} - \not{\varepsilon}\not{k}\right\}\not{p}\not{k}'\left\{2\underbrace{(p\cdot\varepsilon)}_{0} - \not{p}\not{\varepsilon}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
= & -16\, m^2 (\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) \\
& -4(k\cdot\varepsilon')(k'\cdot\varepsilon)\,\mathrm{Tr}\left[\not{\varepsilon}'\left\{2\underbrace{(p\cdot\varepsilon)}_{0} - \not{\varepsilon}\not{p}\right\}\not{p}\right] - 2(k\cdot\varepsilon')\,\mathrm{Tr}\left[\not{\varepsilon}'\not{p}\not{k}'\not{p}\right] \\
& -16(\varepsilon\cdot\varepsilon')(p\cdot k')\left[(k\cdot\varepsilon')\underbrace{(p\cdot\varepsilon)}_{0} - (k\cdot p)(\varepsilon\cdot\varepsilon') + \underbrace{(k\cdot\varepsilon)}_{0}\underbrace{(p\cdot\varepsilon')}_{0}\right] \\
& +2m^2(\varepsilon\cdot\varepsilon')\,\mathrm{Tr}\left[\not{k}\not{\varepsilon}'\not{k}'\not{\varepsilon}\right] + \mathrm{Tr}\left[\not{\varepsilon}\not{k}\not{p}\not{k}'\not{p}\not{\varepsilon}\right]
\end{aligned}
$$

$$
\begin{aligned}
= & -16m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) + 4(k\cdot\varepsilon')(k'\cdot\varepsilon)\,\mathrm{Tr}\left[\not{\varepsilon}'\not{\varepsilon}\underbrace{\not{p}\not{p}}_{m^2}\right] \\
& -8(k\cdot\varepsilon')\left[\underbrace{(p\cdot\varepsilon')}_{0}(p\cdot k') - \underbrace{(k'\cdot\varepsilon')}_{0}p^2 + \underbrace{(p\cdot\varepsilon')}_{0}(p\cdot k')\right] \\
& +16(\varepsilon\cdot\varepsilon')^2(p\cdot k)(p\cdot k') + 8m^2(\varepsilon\cdot\varepsilon')\left[(k\cdot\varepsilon')(k'\cdot\varepsilon) - (k\cdot k')(\varepsilon\cdot\varepsilon') + \underbrace{(k\cdot\varepsilon)}_{0}\underbrace{(k'\cdot\varepsilon')}_{0}\right] \\
& + \mathrm{Tr}\left[\not{k}\not{p}\not{k}'\not{p}\underbrace{\not{\varepsilon}\not{\varepsilon}}_{-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
= & -16m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) + 4m^2(k\cdot\varepsilon')(k'\cdot\varepsilon)\,\mathrm{Tr}\left[\not{\varepsilon}'\not{\varepsilon}\right] \\
& +16(\varepsilon\cdot\varepsilon')^2(p\cdot k)(p\cdot k') + 8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) - 8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k') \\
& -4\left[(p\cdot k)(p\cdot k') - (k\cdot k')p^2 + (p\cdot k)(p\cdot k')\right]
\end{aligned}
$$

$$= -16m^2\cancel{(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')}(k'\cdot\varepsilon) + 16m^2\cancel{(\varepsilon\cdot\varepsilon')}(k\cdot\varepsilon')(k'\cdot\varepsilon)$$

$$+16(\varepsilon\cdot\varepsilon')^2(p\cdot k)(p\cdot k') + 8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) - 8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k')$$

$$-4(p\cdot k)(p\cdot k') + 4(k\cdot k')m^2 - 4(p\cdot k)(p\cdot k')$$

$$= 16(\varepsilon\cdot\varepsilon')^2(p\cdot k)(p\cdot k') - 8(p\cdot k)(p\cdot k')$$

$$+8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) - 8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k') + 4m^2(k\cdot k')$$

A - 2

$$= \mathrm{Tr}\left[\not\varepsilon'\not k\not\varepsilon\not p\not\varepsilon'\not k'\not\varepsilon\not k\right]$$

$$= \mathrm{Tr}\left[\{2(k\cdot\varepsilon') - \not k\not\varepsilon'\}\not\varepsilon\not p\not\varepsilon'\not k'\not\varepsilon\not k\right]$$

$$= 2(k\cdot\varepsilon')\,\mathrm{Tr}\left[\not\varepsilon\not p\not\varepsilon'\not k'\not\varepsilon\not k\right] - \mathrm{Tr}\left[\not k\not\varepsilon'\not\varepsilon\not p\not\varepsilon'\not k'\not\varepsilon\not k\right]$$

$$= 2(k\cdot\varepsilon')\,\mathrm{Tr}\left[\not\varepsilon\not p\not\varepsilon'\not k'\{2\underbrace{(k\cdot\varepsilon)}_{0} - \not k\not\varepsilon\}\right] - \mathrm{Tr}\left[\not\varepsilon'\not\varepsilon\not p\not\varepsilon'\not k'\not\varepsilon\underbrace{\not k\not k}_{0}\right]$$

$$= -2(k\cdot\varepsilon')\,\mathrm{Tr}\left[\not\varepsilon\not p\not\varepsilon'\not k'\not k\not\varepsilon\right]$$

$$= -2(k\cdot\varepsilon')\,\mathrm{Tr}\left[\not p\not\varepsilon'\not k'\not k\underbrace{\not\varepsilon\not\varepsilon}_{-1}\right]$$

$$= 8(k\cdot\varepsilon')\left[\underbrace{(p\cdot\varepsilon')}_{0}(k\cdot k') - (p\cdot k')(k\cdot\varepsilon') + (p\cdot k)\underbrace{(k'\cdot\varepsilon')}_{0}\right]$$

$$= -8(k\cdot\varepsilon')^2(p\cdot k')$$

Ⓐ-③

$= \mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}'\not{k}'\not{\varepsilon}\not{k}']$

$= \mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}'\not{k}'\{2(k'\cdot\varepsilon)-\not{k}'\not{\varepsilon}\}]$

$= 2(k'\cdot\varepsilon)\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}'\not{k}'] - \mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}'\underbrace{\not{k}'\not{k}'}_{0}\not{\varepsilon}]$

$= 2(k'\cdot\varepsilon)\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\{2(\underbrace{k'\cdot\varepsilon'}_{0})-\not{k}'\not{\varepsilon}'\}]$

$= -2(k'\cdot\varepsilon)\mathrm{Tr}[\not{k}\not{\varepsilon}\not{p}\not{k}'\underbrace{\not{\varepsilon}'\not{\varepsilon}'}_{-1}]$

$= 8(k'\cdot\varepsilon)[(\underbrace{k\cdot\varepsilon}_{0})(p\cdot k')-(p\cdot k)(k'\cdot\varepsilon)+(k\cdot k')(\underbrace{p\cdot\varepsilon}_{0})]$

$= -8(k'\cdot\varepsilon)^2(p\cdot k)$

よって、

$$\begin{aligned}Ⓐ = &16(\varepsilon\cdot\varepsilon')^2(p\cdot k)(p\cdot k') - 8(p\cdot k)(p\cdot k') \\ &+8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) - 8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k') + 4m^2(k\cdot k') \\ &-8(k\cdot\varepsilon')^2(p\cdot k') + 8(k'\cdot\varepsilon)^2(p\cdot k)\end{aligned}$$

また、

$Ⓑ = m^2\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}'\not{k}'\not{\varepsilon}]$

$= m^2\mathrm{Tr}[\not{\varepsilon}'\not{k}\{2(\varepsilon\cdot\varepsilon')-\not{\varepsilon}'\not{\varepsilon}\}\not{k}'\not{\varepsilon}]$

$= 2m^2\underbrace{\mathrm{Tr}}_{(\varepsilon\cdot\varepsilon')}[\not{\varepsilon}'\not{k}\not{k}'\not{\varepsilon}] - m^2\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{\varepsilon}\not{k}'\not{\varepsilon}]$

$= 8m^2(\varepsilon\cdot\varepsilon')[(k\cdot\varepsilon')(k'\cdot\varepsilon)-(\underbrace{k'\cdot\varepsilon'}_{0})(k\cdot\varepsilon)+(\varepsilon\cdot\varepsilon')(k\cdot k')]$

$- m^2\mathrm{Tr}[\{2(k\cdot\varepsilon')-\not{k}\not{\varepsilon}'\}\not{\varepsilon}'\not{\varepsilon}\not{k}'\not{\varepsilon}]$

$$= 8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) + 8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k')$$

$$- m^2\,\mathrm{Tr}\left[\{2(k\cdot\varepsilon') - \not{k}\not{\varepsilon}'\}\not{\varepsilon}'\not{\varepsilon}\{2(k'\cdot\varepsilon) - \not{\varepsilon}\not{k}'\}\right]$$

$$= 8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) + 8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k')$$

$$- 4m^2(k\cdot\varepsilon')(k'\cdot\varepsilon)\,\mathrm{Tr}[\not{\varepsilon}'\not{\varepsilon}] + 2m^2(k\cdot\varepsilon')\,\mathrm{Tr}[\not{\varepsilon}'\underbrace{\not{\varepsilon}\not{\varepsilon}}_{-1}\not{k}']$$

$$+ 2m^2(k'\cdot\varepsilon)\,\mathrm{Tr}[\not{k}\underbrace{\not{\varepsilon}'\not{\varepsilon}'}_{-1}\not{\varepsilon}] - m^2\,\mathrm{Tr}[\not{k}\underbrace{\not{\varepsilon}'\not{\varepsilon}'}_{-1}\underbrace{\not{\varepsilon}\not{\varepsilon}}_{-1}\not{k}']$$

$$= 8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) + 8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k')$$

$$- 16m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) - 8m^2(k\cdot\varepsilon')\underbrace{(k'\cdot\varepsilon')}_{0}$$

$$- 8m^2(k'\cdot\varepsilon)\underbrace{(k\cdot\varepsilon)}_{0} - 4m^2(k\cdot k')$$

$$= -8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon) + 8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k') - 4m^2(k\cdot k')\,]$$

よって.

Ⓐ+Ⓑ

$$= 16(\varepsilon\cdot\varepsilon')^2(p\cdot k)(p\cdot k') - 8(p\cdot k)(p\cdot k')$$

$$+ \cancel{8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon)} - \cancel{8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k')} + \cancel{4m^2(k\cdot k')}$$

$$- 8(k\cdot\varepsilon')^2(p\cdot k') + 8(k'\cdot\varepsilon)^2(p\cdot k)$$

$$- \cancel{8m^2(\varepsilon\cdot\varepsilon')(k\cdot\varepsilon')(k'\cdot\varepsilon)} + \cancel{8m^2(\varepsilon\cdot\varepsilon')^2(k\cdot k')} - \cancel{4m^2(k\cdot k')}$$

$$= 8\{2(\varepsilon\cdot\varepsilon')^2 - 1\}(p\cdot k)(p\cdot k') - 8(k\cdot\varepsilon')^2(p\cdot k') + 8(k'\cdot\varepsilon)^2(p\cdot k)$$

$$\therefore\ \frac{1}{2}\sum_{s,s'} T_1 T_2^* = \frac{\{2(\varepsilon\cdot\varepsilon')^2-1\}(p\cdot k)(p\cdot k') - (k\cdot\varepsilon')^2(p\cdot k') + (k'\cdot\varepsilon)^2(p\cdot k)}{E_p E_{p'}}$$

この複素共役をとると、右辺は不変であり、

$$\left(\frac{1}{2}\sum_{s,s'} T_1 T_2^*\right) = \frac{1}{2}\sum_{s,s'} T_1^* T_2$$ となるので、 $$\frac{1}{2}\sum_{s,s'} T_1 T_2^* = \frac{1}{2}\sum_{s,s'} T_1^* T_2$$

これらの結果より、

$$\frac{1}{2}\sum_{s,s'}|S|^2 = \frac{e^4}{4V^4\omega_k\omega_{k'}} \times \frac{1}{2}\sum_{s,s'}\left[\frac{|T_1|^2}{4(p\cdot k)^2} + \frac{|T_2|^2}{4(p\cdot k')^2} + \frac{\overset{2T_1T_2^*}{T_1T_2^* + T_1^*T_2}}{4(p\cdot k)(p\cdot k')}\right]$$

$$\times(2\pi)^8\left[\delta^{(4)}(p+k-p'-k')\right]^2$$

$$= \frac{e^4}{4V^4\omega_k\omega_{k'}}\left[\frac{(p\cdot k)[2(k\cdot\varepsilon')^2+(p\cdot k')]}{4E_pE_{p'}(p\cdot k)^2} - \frac{(p\cdot k')[2(k'\cdot\varepsilon)^2-(p\cdot k)]}{4E_pE_{p'}(p\cdot k')^2}\right.$$

$$\left.+\frac{\{2(\varepsilon\cdot\varepsilon')^2-1\}(p\cdot k)(p\cdot k') - (k\cdot\varepsilon')^2(p\cdot k') + (k'\cdot\varepsilon)^2(p\cdot k)}{2E_pE_{p'}(p\cdot k)(p\cdot k')}\right]$$

$$\times\lim_{t,V\to\infty}(2\pi)^4\, tV\,\delta^{(4)}(p+k-p'-k')$$

$$= \lim_{t,V\to\infty} \frac{e^4}{16V^3 \omega_k \omega_{k'} E_p E_{p'}} \times (2\pi)^4 t \, \delta^{(4)}(p+k-p'-k')$$

$$\times \left[\frac{2(\cancel{k\cdot\varepsilon'})^2 + (p\cdot k)}{(p\cdot k)} - \frac{2(\cancel{k'\cdot\varepsilon})^2 - (p\cdot k)}{(p\cdot k')} \right.$$

$$\left. + 2\{2(\varepsilon\cdot\varepsilon') - 1\} - \frac{2(\cancel{k\cdot\varepsilon'})^2}{(p\cdot k)} + \frac{2(\cancel{k'\cdot\varepsilon})^2}{(p\cdot k')} \right]$$

$$\boxed{\begin{aligned} p\cdot k &= m\omega_k \\ p\cdot k' &= m\omega_{k'} \end{aligned}}$$

$$= \lim_{t,V\to\infty} \frac{e^4}{16V^3 \omega_k \omega_{k'} E_p E_{p'}} \times (2\pi)^4 t \, \delta^{(4)}(p+k-p'-k')$$

$$\times \left[\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} + 4(\varepsilon\cdot\varepsilon')^2 - 2 \right]$$

さらに、微分断面積を求めるため、tで割って$\underbrace{\text{運動量}}_{\text{終状態の}}$で和をとる。

ただし、まだ偏極和はとらない。

$|k'| = k'$としている。よって$k' = \omega_{k'}$

$$\left(\frac{d\sigma}{d\Omega}\right)_{Lab,pol} = \frac{\cancel{V}}{\underbrace{v_{rel}}_{1}} \cdot \cancel{V} \int_0^\infty \frac{dk' k'^2}{(2\pi)^3} \times \cancel{V} \int \underbrace{\frac{d^3p'}{(2\pi)^3}}_{\text{先に実行}} \frac{e^4}{16\cancel{V^3} \omega_k \omega_{k'} E_p E_{p'}}$$

$$\times \left[\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} + 4(\varepsilon\cdot\varepsilon')^2 - 2 \right] \times (2\pi)^4 \delta^{(4)}(p+k-p'-k')$$

$$= \int_0^\infty \frac{d\omega_{k'}}{(2\pi)^3} \omega_{k'}^2 \cdot \frac{e^4}{16\omega_k \omega_{k'} E_p E_{p'}} \left[\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} + 4(\varepsilon\cdot\varepsilon')^2 - 2 \right]$$

$$\times (2\pi)\, \delta(m+\omega_k - E_{p'} - \omega_{k'})$$

$E_{p'}$を消すため、$E_{p'}+\omega_{k'}$に変数変換すると.

$$d\omega_{k'} = \frac{d\omega_{k'}}{d(\omega_{k'}+E_{p'})} d(\omega_{k'}+E_{p'})$$

$$E_{p'}^2 = \mathbb{P}'^2 + m^2 = (\mathbb{k}-\mathbb{k}')^2 + m^2$$

$\leftarrow$ $\mathbb{P}=0$より. 運動量保存は

$\mathbb{k} = \mathbb{k}' + \mathbb{P}'$

$\therefore \mathbb{P}' = \mathbb{k} - \mathbb{k}'$

$$= \omega_k^2 + \omega_{k'}^2 - 2\omega_k\omega_{k'}\cos\theta + m^2.$$

$$\Leftrightarrow 2E_{p'}\frac{dE_{p'}}{d\omega_{k'}} = 2\omega_{k'} - 2\omega_k\cos\theta$$

$$\Leftrightarrow \boxed{\frac{dE_{p'}}{d\omega_{k'}} = \frac{\omega_{k'} - \omega_k\cos\theta}{E_{p'}}}$$

また.. $\frac{d\omega_{k'}}{d\omega_{k'}} = 1$より.

$$\frac{d(E_{p'}+\omega_{k'})}{d\omega_{k'}} = 1 + \frac{\omega_{k'} - \omega_k\cos\theta}{E_{p'}}$$

$$= \frac{\boxed{E_{p'}+\omega_{k'}} - \omega_k\cos\theta}{E_{p'}}$$

$$= \frac{m + \omega_k - \omega_k\cos\theta}{E_{p'}}$$

$$= \frac{m + \omega_k(1-\cos\theta)}{E_{p'}}$$

$$= \frac{m\omega_k}{E_{p'}\omega_{k'}}$$

$$\boxed{\omega_{k'} = \frac{m\omega_k}{m+\omega_k(1-\cos\theta)}}$$

$$\therefore d\omega_{k'} = \frac{E_{p'}\omega_{k'}}{m\omega_k} d(E_{p'}+\omega_{k'})$$

以上より.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Lab,pol}} = \int_m^\infty d(E_{p'}+\omega_{k'})\,\frac{\cancel{E_{p'}}\,\cancel{\omega_{k'}}}{m\,\omega_k}\cdot\frac{\omega_{k'}^2\,e^4}{(2\pi)^2\cdot 16\,\omega_k\,\cancel{\omega_{k'}}\,\underbrace{E_p}_{m}\,\cancel{E_{p'}}}$$

$$\times\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')^2-2\right]\times\delta(m+\omega_k-E_{p'}-\omega_{k'})$$

$$= \frac{e^4\,\omega_{k'}^2}{(4\pi)^2\cdot 4m^2\,\omega_k^2}\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')^2-2\right]$$

$$= \underbrace{\frac{\alpha^2}{4m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')^2-2\right]}$$

Klein-Nishina formula

$$\boxed{\sum_\lambda \varepsilon_i(\lambda,\mathbb{k})\,\varepsilon_j(\lambda,\mathbb{k}) = \delta_{ij}-\frac{k_i k_j}{\mathbb{k}^2}}$$

始状態の偏極の平均、終状態の偏極和をとる。

$$\frac{1}{2}\sum_{\lambda\lambda'}(\varepsilon\cdot\varepsilon')^2 = \frac{1}{2}\sum_{\lambda\lambda'}\varepsilon_i(\lambda,\mathbb{k})\,\varepsilon_j(\lambda,\mathbb{k})\,\varepsilon_i(\lambda',\mathbb{k}')\,\varepsilon_j(\lambda',\mathbb{k}')$$

$$= \frac{1}{2}\left(\delta_{ij}-\frac{k_i k_j}{\mathbb{k}^2}\right)\left(\delta_{ij}-\frac{k'_i k'_j}{\mathbb{k}'^2}\right)$$

$$= \frac{1}{2}\left(3-\underbrace{\frac{\mathbb{k}^2}{\mathbb{k}^2}}_{1}-\underbrace{\frac{\mathbb{k}'^2}{\mathbb{k}'^2}}_{1}+\frac{(\mathbb{k}\cdot\mathbb{k}')^2}{\mathbb{k}^2\,\mathbb{k}'^2}\right)$$

$$= \frac{1}{2}\left(1+\cos^2\theta\right)$$

を用いて.

$$\left(\frac{d\sigma}{d\Omega}\right)_{Lab} = \frac{1}{2}\sum_{\lambda,\lambda'} \frac{\alpha^2}{4m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2 \left[\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} + 4(\varepsilon\cdot\varepsilon')^2 - 2\right]$$

$$= \frac{\alpha^2}{4m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2 \left[2\frac{\omega_{k'}}{\omega_k} + 2\frac{\omega_k}{\omega_{k'}} + 2(1+\cos^2\theta) - 4\right]$$

$$= \frac{\alpha^2}{2m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2 \left[\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} - (1-\cos^2\theta)\right]$$

忘れるな

$$\boxed{\frac{1}{2}\sum_{\lambda}\sum_{\lambda'} = \frac{1}{2}\cdot 2\cdot 2 = 2}$$

$$= \frac{\alpha^2}{2m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2 \left[\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} - \sin^2\theta\right] \quad //$$

最後に、$\omega_{k'} = \dfrac{m\omega_k}{m+\omega_k(1-\cos\theta)}$ を用いて書き換えると、

。 $$\frac{\omega_{k'}}{\omega_k} = \frac{m}{m+\omega_k(1-\cos\theta)}$$

。 $$\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} = \frac{m}{m+\omega_k(1-\cos\theta)} + \frac{m+\omega_k(1-\cos\theta)}{m}$$

$$= \frac{m^2 + \{m+\omega_k(1-\cos\theta)\}^2}{m\{m+\omega_k(1-\cos\theta)\}}$$

$$= \frac{m^2+m^2+2m\omega_k(1-\cos\theta)+\omega_k^2(1-\cos\theta)^2}{m\{m+\omega_k(1-\cos\theta)\}}$$

$$= \frac{2m\{m+\omega_k(1-\cos\theta)\}+\omega_k^2(1-\cos\theta)^2}{m\{m+\omega_k(1-\cos\theta)\}}$$

$$= 2 + \frac{\omega_k^2(1-\cos\theta)^2}{m\{m+\omega_k(1-\cos\theta)\}}$$

以上より

$$\left(\frac{d\sigma}{d\Omega}\right)_{Lab} = \frac{\alpha^2}{2m^2}\cdot\left[\frac{m}{m+\omega_k(1-\cos\theta)}\right]^2\left[2+\frac{\omega_k^2(1-\cos\theta)^2}{m\{m+\omega_k(1-\cos\theta)\}}-\sin^2\theta\right]$$

$$= \frac{\alpha^2}{2}\cdot\frac{1}{[m+\omega_k(1-\cos\theta)]^2}\cdot\left[\frac{\omega_k^2(1-\cos\theta)^2}{m\{m+\omega_k(1-\cos\theta)\}}+1+\cos^2\theta\right]$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{Lab} = \frac{\alpha^2}{2}\cdot\frac{1}{[m+\omega_k(1-\cos\theta)]^2}\left[\frac{\omega_k^2(1-\cos\theta)^2}{m\{m+\omega_k(1-\cos\theta)\}}+1+\cos^2\theta\right]}$$

$\boxed{\text{Thomson 散乱}}$

光子のエネルギーがターゲット電子より非常に小さいとき、

$\omega_k \ll m$ より $\frac{\omega_k}{m} \ll 1$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\alpha^2}{2}\cdot\frac{1}{m^2\left[1+\underbrace{\frac{\omega_k}{m}(1-\cos\theta)}_{0}\right]^2}\left[\frac{\omega_k^2(1-\cos\theta)^2}{\underbrace{m^2\left\{1+\frac{\omega_k}{m}(1-\cos\theta)\right\}}_{0}}+1+\cos^2\theta\right]$$

$$\fallingdotseq \frac{\alpha^2}{2m^2}(1+\cos^2\theta) = r_0^2\,\frac{1+\cos^2\theta}{2} \qquad \underbrace{r_0 = \frac{e^2}{4\pi m c^2} = \frac{e^2}{4\pi\hbar c}\cdot\frac{\hbar}{mc}}_{\text{古典電子半径}}$$

全断面積は

$$\sigma = \int d\Omega\, r_0^2\,\frac{1+\cos^2\theta}{2} = \int_{-1}^{1}dz\int_0^{2\pi}d\varphi\, r_0^2\,\frac{1+z^2}{2}$$

$$= \pi r_0^2\left[z+\frac{z^3}{3}\right]_{-1}^{1} = \frac{8\pi}{3}r_0^2 \rightarrow \text{Thomson の断面積}$$