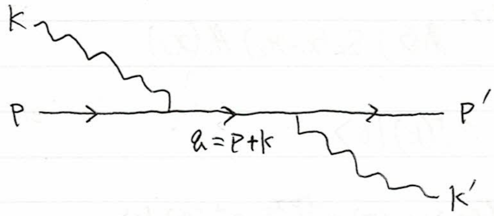
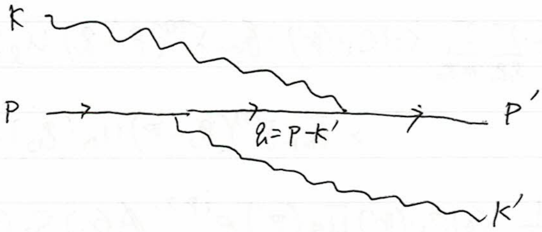
Compton scattering.





$$S=\sum_{n=0}^{\infty}\frac{(ie)^n}{n!}\iint\cdots\int d^4x_1\cdots d^4x_n\, T\{N[\bar\psi A\!\!\!/\psi]_{x_1}\cdots N[\bar\psi A\!\!\!/\psi]_{x_n}\}$$

相互作用の2次では.

$$S_e^{(2)}=\frac{(ie)^2}{2}\int d^4x_1\int d^4x_2\ T\{N[\bar\psi A\!\!\!/\psi]_{x_1}N[\bar\psi A\!\!\!/\psi]_{x_2}\}$$

運動量Pの電子に 運動量kの 光子を当てる。

始状態 $|i\rangle=a_r^+(p)\,c_\lambda^+(k)|0\rangle$. 終状態 $|f\rangle=a_s^+(p')\,c_{\lambda'}^+(k')|0\rangle$

$\langle f|S_e^{(2)}|i\rangle$の計算手順.

まず、内線部分の縮約をとると、

$$S_e^{(2)}=\frac{(ie)^2}{2}\int d^4x_1\int d^4x_2\Big(T\{N[\bar\psi A\!\!\!/\overbrace{\psi]_{x_1}N[\bar\psi}A\!\!\!/\psi]_{x_2}\}+T\{N[\overbrace{\bar\psi A\!\!\!/\psi]_{x_1}N[\bar\psi A\!\!\!/\psi}]_{x_2}\}\Big)$$

$$=-\frac{e^2}{2}\int d^4x_1\int d^4x_2\cdot 2\,N[\bar\psi A\!\!\!/\overbrace{\psi]_{x_1}N[\bar\psi}\,A\!\!\!/\psi]_{x_2}$$

$$=-e^2\int d^4x_1\int d^4x_2\,\bar\psi(x_1)A\!\!\!/(x_1)\,S_F(x_1-x_2)\,A\!\!\!/(x_2)\psi(x_2)$$

次に、フェルミオンの外線の縮約をとると、

$$\langle f|S_e^{(2)}|i\rangle=\langle 0|c_{\lambda'}(k')\overbrace{a_s(p')\,(-ie^2)\int d^4x_1\int d^4x_2\ \bar\psi_-}(x_1)A\!\!\!/(x_1)\,S_F(x_1-x_2)A\!\!\!/(x_2)\overbrace{\psi_+(x_2)\,|a_r^+}(p)c_\lambda(k)|0\rangle$$

$$=-ie^2\int d^4x_1\int d^4x_2\langle 0|c_{\lambda'}(k')\overbrace{a_s(p')\Big(\sum_{\ell,q_1}\frac{1}{\sqrt V}a_\ell^+}(q_1)\bar u_\ell(q_1)e^{iq_1x_1}\Big)A\!\!\!/(x_1)\ S_F(x_1-x_2)A\!\!\!/(x_2)$$

$$\times\Big(\sum_{m,q_2}\frac{1}{\sqrt V}\overbrace{a_m^+(q_2)u_m(q_2)e^{-iq_2x_2}\Big)a_r^+}(p)\,c_\lambda(k)|0\rangle$$

$$= -ie^2 \int d^4x_1 \int d^4x_2 \frac{1}{V} \cdot \sum_{l,q_1} \sum_{m,q_2} \langle 0 | C_{\lambda'}(\mathbb{k}') \cdot \delta_{sl} \delta^{(3)}(\mathbb{p}' - q_1) \bar{u}_l(q_1) e^{iq_1x_1} \not{A}(x_1) S_F(x_1 - x_2) \not{A}(x_2)$$

$$\times \delta_{mr} \delta^{(3)}(q_2 - \mathbb{p}) u_m(q_2) e^{-iq_2x_2} \; C_\lambda^\dagger(k) | 0 \rangle$$

$$= -ie^2 \int d^4x_1 \int d^4x_2 \frac{1}{V} \cdot \langle 0 | C_{\lambda'}(\mathbb{k}') \bar{u}_s(\mathbb{p}') e^{ip'x_1} \not{A}(x_1) S_F(x_1 - x_2) \not{A}(x_2) u_r(\mathbb{p}) e^{-ipx_2} \; C_\lambda^\dagger(\mathbb{k}) | 0 \rangle$$

$$= \frac{-ie^2}{V} \int d^4x_1 \int d^4x_2 \; \bar{u}_s(\mathbb{p}') \gamma^\mu S_F(x_1 - x_2) \gamma^\nu u_r(\mathbb{p}) e^{ip'x_1 - ipx_2} \langle 0 | C_{\lambda'}(\mathbb{k}') A_\mu(x_1) A_\nu(x_2) C_\lambda^\dagger(\mathbb{k}) | 0 \rangle$$

最後に光子の外線をつなぐと、

$$\langle 0 | \overbrace{C_{\lambda'}(\mathbb{k}') A_\mu(x_1)} \overbrace{A_\nu(x_2) C_\lambda^\dagger(\mathbb{k})} | 0 \rangle + \langle 0 | \overbrace{C_{\lambda'}(\mathbb{k}') A_\nu(x_2)} \overbrace{A_\mu(x_1) C_\lambda^\dagger(\mathbb{k})} | 0 \rangle$$

$$= \langle 0 | C_{\lambda'}(\mathbb{k}') \left(\sum_{\lambda_1} \int \frac{d^3\mathbb{k}_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_1}}} \varepsilon_\mu(\lambda_1, k_1) C_{\lambda_1}^\dagger(\mathbb{k}_1) e^{ik_1x_1} \right) \left(\sum_{\lambda_2} \int \frac{d^3\mathbb{k}_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_2}}} \varepsilon_\nu(\lambda_2, k_2) C_{\lambda_2}(k_2) e^{-ik_2x_2} \right) C_\lambda^\dagger(\mathbb{k}) | 0 \rangle$$

$$+ \langle 0 | C_{\lambda'}(\mathbb{k}') \left(\sum_{\lambda_2} \int \frac{d^3\mathbb{k}_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_2}}} \varepsilon_\nu(\lambda_2, \mathbb{k}_2) C_{\lambda_2}^\dagger(\mathbb{k}_2) e^{ik_2x_2} \right) \left(\sum_{\lambda_1} \int \frac{d^3\mathbb{k}_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_1}}} \varepsilon_\mu(\lambda_1, k_1) C_\lambda(\mathbb{k}_1) e^{-ik_1x_1} \right) C_\lambda^\dagger(\mathbb{k}) | 0 \rangle$$

$$= \sum_{\lambda_1\lambda_2} \int \frac{d^3\mathbb{k}_1 d^3\mathbb{k}_2}{(2\pi)^{3\cdot 2}} \frac{1}{2\sqrt{\omega_{k_1}\omega_{k_2}}} \Big[(2\pi)^3 \delta_{\lambda'\lambda_1} \delta^{(3)}(\mathbb{k}' - \mathbb{k}_1) \varepsilon_\mu(\lambda_1, k_1) e^{ik_1x_1} \times (2\pi)^3 \delta_{\lambda_2\lambda} \delta^{(3)}(\mathbb{k}_2 - \mathbb{k}) \varepsilon_\nu(\lambda_2, k_2) e^{-ik_2x_2}$$

$$+ \qquad + (2\pi)^3 \delta_{\lambda'\lambda_2} \delta^{(3)}(\mathbb{k}' - \mathbb{k}_2) \varepsilon_\nu(\lambda_2, \mathbb{k}_2) e^{ik_2x_2} \times (2\pi)^3 \delta_{\lambda_1\lambda} \delta^{(3)}(\mathbb{k}_1 - \mathbb{k}) \varepsilon_\mu(\lambda_1, k_1) e^{-ik_1x_1} \Big]$$

$$= \frac{1}{2\sqrt{\omega_k \omega_{k'}}} \Big[\varepsilon_\mu(\lambda', \mathbb{k}') \varepsilon_\nu(\lambda, k) e^{ik'x_1} e^{-ikx_2} + \varepsilon_\mu(\lambda, k) \varepsilon_\nu(\lambda', k') e^{-ikx_1} e^{ik'x_2} \Big]$$

これにより.

$$\langle f | S_e^{(2)} | i \rangle = -\frac{ie^2}{V}\int d^4x_1 \int d^4x_2\, \bar{u}_s(p')\gamma^\mu S_F(x_1-x_2)\gamma^\nu u_r(p)\, e^{ip'x_1} e^{-ipx_2}$$

$$\times \frac{1}{2\sqrt{\omega_k\omega_{k'}}}\Big[\underbrace{\varepsilon_\mu(\lambda',k')\varepsilon_\nu(\lambda,k)e^{ik'x_1}e^{-ikx_2}}_{ⓐ} + \underbrace{\varepsilon_\mu(\lambda,k)\varepsilon_\nu(\lambda',k')e^{-ikx_1}e^{ik'x_2}}_{ⓑ}\Big]$$

$$=: S_a + S_b.$$

$$S_a = -\frac{ie^2}{V}\int d^4x_1\int d^4x_2 \frac{1}{2\sqrt{\omega_k\omega_{k'}}}\bar{u}_s(p')\gamma^\mu\int\frac{d^4q}{(2\pi)^4}\frac{\not{q}+m}{q^2-m^2+i\epsilon}e^{-iq(x_1-x_2)}\gamma^\nu u_r(p)\varepsilon_\mu(\lambda',k')\varepsilon_\nu(\lambda,k)$$

$$\times e^{ip'x_1}e^{-ipx_2}e^{ik'x_1}e^{-ikx_2}$$

$$= -\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\int d^4x_1\int d^4x_2\int\frac{d^4q}{(2\pi)^4}\bar{u}_s(p')\gamma^\mu\frac{\not{q}+m}{q^2-m^2+i\epsilon}\gamma^\nu u_r(p)\varepsilon_\mu(\lambda',k')\varepsilon_\nu(\lambda,k)$$

$$\times e^{-i(q-p'-k')x_1}e^{-i(p+k-q)x_2}$$

$$= -\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\int\frac{d^4q}{(2\pi)^4}\bar{u}_s(p')\not{\epsilon}'\frac{\not{q}+m}{q^2-m^2+i\epsilon}\not{\epsilon}\, u_r(p)\times(2\pi)^4\delta^{(4)}(q-p'-k')\times(2\pi)^4\delta^{(4)}(p+k-q)$$

$$= -\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\bar{u}_s(p')\not{\epsilon}'\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i\epsilon}\not{\epsilon}\, u_r(p)\times(2\pi)^4\delta^{(4)}(p+k-p'-k')$$

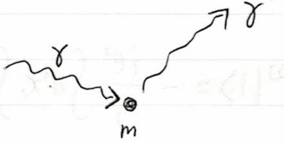
$$S_b = -\frac{ie^2}{V}\int d^4x_1\int d^4x_2\frac{1}{2\sqrt{\omega_k\omega_{k'}}}\bar{u}_s(p')\not{\epsilon}\int\frac{d^4q}{(2\pi)^4}\frac{q+m}{q^2-m^2+i\epsilon}e^{-iq(x_1-x_2)}\not{\epsilon}'u_r(p)\,e^{ip'x_1}e^{-ipx_2}e^{-ikx_1}e^{ik'x_2}$$

$$= -\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\int d^4x_1\int d^4x_2\int\frac{d^4q}{(2\pi)^4}\bar{u}_s(p')\not{\epsilon}\frac{\not{q}+m}{q^2-m^2+i\epsilon}\not{\epsilon}'u_r(p)\,e^{-i(q-p'-k)x_1}e^{-i(p-k'-q)x_2}$$

$$= -\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\int\frac{d^4q}{(2\pi)^4}\bar{u}_s(p')\not{\epsilon}\frac{\not{q}+m}{q^2-m^2+i\epsilon}\not{\epsilon}'u_r(p)\times(2\pi)^4\delta^{(4)}(q-p'-k)\cdot(2\pi)^4\delta^{(4)}(p-k'-q)$$

$$= -\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\bar{u}_s(p')\not{\epsilon}\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2+i\epsilon}\not{\epsilon}'u_r(p)\times(2\pi)^4\delta^{(4)}(p+k-p'-k')$$

光子を静止した電子に入射させるとき、



電子の運動量 $p=(m,0,0,0)$ であるから、

$$\mathbb{P}'=\mathbb{k}-\mathbb{k}'$$

$$E'=\sqrt{(\mathbb{P}')^2+m^2}=\sqrt{(\mathbb{k}-\mathbb{k}')^2+m^2}=\sqrt{m^2+\omega_k^2+\omega_{k'}^2-2\omega_k\omega_{k'}\cos\theta}$$

$$(p+k)^2=p^2+k^2+2p\cdot k=m^2+2p\cdot k$$

$$(p-k')^2=p^2+k^2-2p\cdot k'=m^2-2p\cdot k'$$

よって、

$$\boxed{\begin{aligned}S_a&=-\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\bar{u}_s(p')\not{\epsilon}'\frac{\not{p}+\not{k}+m}{2p\cdot k+i\epsilon}\not{\epsilon}u_r(p)\times(2\pi)^4\delta^{(4)}(p+k-p'-k')\\ S_b&=\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\bar{u}_s(p')\not{\epsilon}\frac{\not{p}-\not{k}'+m}{2p\cdot k'-i\epsilon}\not{\epsilon}'u_r(p)\times(2\pi)^4\delta^{(4)}(p+k-p'-k')\end{aligned}}$$

また、

$k(p+k)=k(p'+k')$ より $p\cdot k=p'\cdot k+k\cdot k'$

$k'(p+k)=k'(p'+k')$ より $p'\cdot k'=p\cdot k'+k\cdot k'$

⇔ $p\cdot k=p\cdot k'+k\cdot k'$

$(p+k)^2=(p'+k')^2$

⇔ $m^2+2p\cdot k=m^2+2p'\cdot k'$

∴ $p\cdot k=p'\cdot k'$

よって実験室系では、

$$p^0k_0=p^0k_0'+\omega_k\omega_{k'}-\omega_k\omega_{k'}\cos\theta$$

⇔ $m\omega_k=m\omega_{k'}+\omega_k\omega_{k'}-\omega_k\omega_{k'}\cos\theta$,

$=\omega_{k'}(m+\omega_k(1-\cos\theta))$

∴ $\boxed{\omega_{k'}=\frac{m\omega_k}{m+\omega_k(1-\cos\theta)}}$

NO.

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実験室系 のとき、さらに

$\varepsilon^\mu = (0, \boldsymbol{\varepsilon})$, $\varepsilon' = (0, \boldsymbol{\varepsilon}')$ であり、Lorentz 条件 より

$k\cdot\varepsilon = -\mathbf{k}\cdot\boldsymbol{\varepsilon} = 0$、 $k'\varepsilon' = -\mathbf{k}'\cdot\boldsymbol{\varepsilon}' = 0$、

また $p\varepsilon = 0$, $p'\varepsilon' = 0$ も成り立つ。

これを用いて、

・ S_a において、

Dirac equation

$(\not p - m)u(p) = 0$

$\bar{u}_s(p')\not\varepsilon'(\not p + \not k + m)\not\varepsilon u_r(p)$ は、

$$\bar{u}_s(p')\not\varepsilon'\not p\not\varepsilon u_r(p) = \bar{u}_s(p')\not\varepsilon'(2p\cdot\varepsilon - \not\varepsilon\not p)u_r(p)$$

$$= \bar{u}_s(p')\not\varepsilon'\cdot 2\underbrace{(p\cdot\varepsilon)}_{0}u_r(p) - \bar{u}_s(p')\not\varepsilon'\not\varepsilon(mu_r(p))$$

$$= -m\bar{u}_s(p')\not\varepsilon'\not\varepsilon u_r(p)$$ より、

$$\bar{u}_s(p')\not\varepsilon'(\not p + \not k + m)\not\varepsilon u_r(p) = \bar{u}_s(p')\not\varepsilon'\not k\not\varepsilon u_r(p)$$

$$\therefore S_a = -\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\cdot\frac{\bar{u}_s(p')\not\varepsilon'\not k\not\varepsilon u_r(p)}{2(p\cdot k) + i\epsilon} \times (2\pi)^4\delta^{(4)}(p+k-p'-k')$$

同様に、

$$S_b = \frac{-ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\frac{\bar{u}_s(p')\not\varepsilon\not k'\not\varepsilon' u_r(p)}{2(p\cdot k') - i\epsilon} \times (2\pi)^4\delta^{(4)}(p+k-p'-k')$$

よって、

$$S = -\frac{ie^2}{2V\sqrt{\omega_k\omega_{k'}}}\cdot\left[\frac{\bar{u}_s(p')\not\varepsilon'\not k\not\varepsilon u_r(p)}{2(p\cdot k)+i\epsilon} + \frac{\bar{u}_s(p')\not\varepsilon\not k'\not\varepsilon' u_r(p)}{2(p\cdot k')-i\epsilon}\right] \times (2\pi)^4\delta^{(4)}(p+k-p'-k')$$

$$= -\frac{ie^2}{2V\sqrt{\omega_k\omega_k}}\left[\frac{T_a}{2(p\cdot k)+i\epsilon} + \frac{T_b}{2(p\cdot k')-i\epsilon}\right] \times (2\pi)^4\delta^{(4)}(p+k-p'-k')$$

これより

$$|S|^2 = \frac{e^4}{4V^2\omega_k\omega_{k'}}\left[\frac{|T_a|^2}{4(p\cdot k)^2} + \frac{|T_b|^2}{4(p\cdot k')^2} + \frac{T_aT_b^* + T_a^*T_b}{4(p\cdot k)(p\cdot k')}\right](2\pi)^8\left[\delta^{(4)}(p+k-p'-k')\right]^2$$

スピン平均と和をとって

$$\frac{1}{2}\sum_{\text{spin}}|T_a|^2 = \frac{1}{2}\sum_r\sum_s\left|\bar{u}_s(p')\not{\varepsilon}'\not{k}\not{\varepsilon}\,u_r(p)\right|^2$$

$$= \frac{1}{2}\sum_r\sum_s \bar{u}_s^*(p')_a \not{\varepsilon}'^*_{ab}\not{k}^*_{bc}\not{\varepsilon}^*_{cd}u_r^*(p)_d\ \ \bar{u}_s(p')_e\not{\varepsilon}'_{ef}\not{k}_{fg}\not{\varepsilon}_{gh}u_r(p)_h$$

$$= \frac{1}{2}\sum_{r,s}\not{\varepsilon}'_{ef}\not{k}_{fg}\not{\varepsilon}_{gh}\,u_r(p)_h\,u_r^*(p)_d\,\not{\varepsilon}^*_{dc}\not{k}^\dagger_{cb}\not{\varepsilon}'^\dagger_{ba}\,\bar{u}_s^\dagger(p')_a\,\bar{u}_s(p')_e$$

$$= \frac{1}{2}\sum_{r,s}\not{\varepsilon}'_{ef}\not{k}_{fg}\not{\varepsilon}_{gh}\,\underline{u_r(p)_h\bar{u}_r(p)_d}\,\not{\varepsilon}_{dc}\not{k}_{cb}\not{\varepsilon}'_{ba}\,\underline{u_s(p')_a\bar{u}_s(p')_e}$$

$$= \frac{1}{8E_pE_{p'}}\mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{\varepsilon}(\not{p}+m)\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p}'+m)\right]$$

$$= \frac{1}{8E_pE_{p'}}\left\{\mathrm{Tr}\Big[\underbrace{\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}\not{k}\not{\varepsilon}'\not{p}'}_{Ⓐ}\Big] + m\,\mathrm{Tr}\Big[\underbrace{\not{\varepsilon}'\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}\not{k}\not{\varepsilon}'}_{Ⓑ}\Big] + m\,\mathrm{Tr}\Big[\underbrace{\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}\not{k}\not{\varepsilon}'\not{p}'}_{Ⓒ}\Big] + m^2\,\mathrm{Tr}\Big[\underbrace{\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}\not{k}\not{\varepsilon}'}_{Ⓓ}\Big]\right\}$$

$$Ⓐ = \mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{\varepsilon}(2p\cdot\varepsilon)\not{k}\not{\varepsilon}'\not{p}'\right] - \mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}\not{p}\not{k}\not{\varepsilon}'\not{p}'\right]$$

$$= 2(p\cdot\varepsilon)\left\{\mathrm{Tr}\left[\not{\varepsilon}'\not{k}(2\underbrace{k\cdot\varepsilon}_{0})\not{\varepsilon}'\not{p}'\right] - \mathrm{Tr}\left[\not{\varepsilon}'\underbrace{\not{k}\not{k}}_{0}\not{\varepsilon}\not{\varepsilon}'\not{p}'\right]\right\} - (\varepsilon\cdot\varepsilon)\,\mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{p}\not{k}\not{\varepsilon}'\not{p}'\right]$$

$$= \cancel{2}\,\mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{p}\not{k}\not{\varepsilon}'\not{p}'\right] = 2(p\cdot k)\,\mathrm{Tr}\left[\not{\varepsilon}'\not{k}\not{\varepsilon}'\not{p}'\right] - \mathrm{Tr}\left[\not{\varepsilon}'\underbrace{\not{k}\not{k}}_{0}\not{p}\not{\varepsilon}'\not{p}'\right]$$

$$= 2(p\cdot k)\left\{2(k\cdot\varepsilon')\,\mathrm{Tr}\left[\not{\varepsilon}'\not{p}'\right] - \mathrm{Tr}\left[\not{\varepsilon}'\not{\varepsilon}'\not{k}\not{p}'\right]\right\}$$

$$= 2(p\cdot k)\left\{8(k\cdot\varepsilon')(p'\cdot\varepsilon') + 4(p'\cdot k)\right\}$$

$$= 8(p\cdot k)\left\{2(k\cdot\varepsilon')^2 + (p\cdot k')\right\}$$

$\mathrm{Tr}[\gamma^\mu\gamma^\nu]a_\mu b_\nu$

$= \mathrm{Tr}[\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}]a_\mu b_\nu$

$= 4a\cdot b.$

$p'-k = p-k'$ より

$\varepsilon'\cdot p' - \varepsilon'\cdot k = 0$

$\longleftarrow$ $\therefore p'\cdot\varepsilon' = k\cdot\varepsilon'$

$$Ⓑ = m\left((p\cdot\varepsilon)\,\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{k}\not{\varepsilon}'] - \mathrm{Tr}[\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}\not{p}\not{k}\not{\varepsilon}']\right)$$

$$= m\left(\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{p}\not{k}\not{\varepsilon}']\right) = -m\,\mathrm{Tr}[\not{k}\not{k}\not{p}] = 0$$

$$Ⓒ = -m\,\mathrm{Tr}[\not{\varepsilon}'\not{k}\not{k}\not{\varepsilon}'\not{p}'] = 0.$$

$$Ⓓ = m^2\,\mathrm{Tr}[\not{\varepsilon}'\not{\varepsilon}'\not{k}\not{\varepsilon}\not{\varepsilon}\not{k}] = 0.$$

よって.

$$\frac{1}{2}\sum_{r,s}|T_a|^2 = \frac{(p\cdot k)\{2(k\cdot\varepsilon')^2+(p\cdot k')\}}{E_pE_{p'}}$$

$p+k = p'+k'$

$\underset{0}{\cancel{p\cdot\varepsilon}} = p'\cdot\varepsilon + k'\cdot\varepsilon$

$\boxed{p'\cdot\varepsilon = -k'\cdot\varepsilon}$

∘ $$\underline{\frac{1}{2}\sum_{r,s}|T_b|^2} = \frac{1}{2}\sum_{r,s}|\bar{u}_s(p')\not{\varepsilon}\not{k}'\not{\varepsilon}'u_r(p)|^2$$

$$= \frac{1}{2}\sum_{r,s}\bar{u}^*_s(p')_a\not{\varepsilon}^*_{ab}\not{k}'^*_{bc}\not{\varepsilon}'^*_{cd}u^*_r(p)_d\,\bar{u}_s(p')_e\not{\varepsilon}_{ef}\not{k}'_{fg}\not{\varepsilon}'_{gh}u_r(p)_h$$

$$= \frac{1}{2}\sum_{r,s}\not{\varepsilon}_{ef}\not{k}'_{fg}\not{\varepsilon}'_{gh}u_r(p)_h\,u^*_r(p)_d\not{\varepsilon}'^\dagger_{dc}\not{k}'^\dagger_{cb}\not{\varepsilon}^\dagger_{ba}\bar{u}^\dagger_s(p)_a\bar{u}_s(p')_e$$

$$= \frac{1}{2}\sum_{r,s}\not{\varepsilon}_{ef}\not{k}'_{fg}\not{\varepsilon}'_{gh}u_r(p)_h\,u^*_r(p)_d(\gamma^0\not{\varepsilon}'\gamma^0)_{dc}(\gamma^0\not{k}'\gamma^0)_{cb}(\gamma^0\not{\varepsilon}\gamma^0)_{ba}(\gamma^0u_s(p'))_a\bar{u}_s(p')_e$$

$$= \frac{1}{2}\sum_{r,s}\not{\varepsilon}_{ef}\not{k}'_{fg}\not{\varepsilon}'_{gh}u_r(p)_h\,\bar{u}^*_r(p)_d\not{\varepsilon}'_{dc}\not{k}'_{cb}\not{\varepsilon}_{ba}u_s(p')_a\bar{u}_s(p')_e$$

$$= \frac{1}{8E_pE_{p'}}\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'(\not{p}+m)\not{\varepsilon}'\not{k}'\not{\varepsilon}(\not{p}'+m)]$$

$$= \frac{1}{8E_pE_{p'}}\left\{\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\underline{\not{p}\not{\varepsilon}'}\not{k}'\not{\varepsilon}\not{p}'] + \underset{0}{m\,\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}'\not{k}'\not{\varepsilon}]} + \underset{0}{m\,\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{\varepsilon}'\not{k}'\not{\varepsilon}\not{p}']} + \underset{0}{m^2\,\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{\varepsilon}'\not{k}'\not{\varepsilon}]}\right\}$$

$$= \frac{1}{8E_pE_{p'}}\left\{2\underset{0}{(p\cdot\varepsilon')}\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{k}'\not{\varepsilon}\not{p}'] - \mathrm{Tr}[\not{\varepsilon}\not{k}'\underline{\not{\varepsilon}'\not{\varepsilon}'}\not{p}\not{k}'\not{\varepsilon}\not{p}']\right\}$$

$$= \frac{1}{8E_pE_{p'}}\mathrm{Tr}[\not{\varepsilon}\not{k}'\underline{\not{p}\not{k}'}\not{\varepsilon}\not{p}'] = \frac{1}{8E_pE_{p'}}\left\{2(p\cdot k')\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}\not{p}'] - \mathrm{Tr}[\not{\varepsilon}\underset{0}{\not{k}'\not{k}'}\not{p}\not{\varepsilon}\not{p}']\right\}$$

$$= \frac{1}{8E_pE_{p'}}\left\{4(p\cdot k')(k'\cdot\varepsilon)\mathrm{Tr}[\not{\varepsilon}\not{p}'] + 2(p\cdot k')\mathrm{Tr}[\not{k}'\not{p}']\right\}$$

$$= \frac{(p\cdot k')\{16(k'\cdot\varepsilon)\cdot(p'\cdot\varepsilon)+8(p'\cdot k')\}}{8E_pE_{p'}} = \underline{\frac{-(p\cdot k')\{2(k'\cdot\varepsilon)^2-(p\cdot k)\}}{E_pE_{p'}}}$$

$$\cdot\ \frac{1}{2}\sum_{r,s} T_a^* T_b = \frac{1}{2}\sum_{r,s} \bar{u}_s^*(p')_a \not\epsilon'^*_{ab} \not k^*_{bc} \not\epsilon^*_{cd}\, u_r^*(p)_d\, \bar{u}_s(p')_e \not\epsilon_{ef} \not k'_{fg} \not\epsilon'_{gh}\, u_r(p)_h$$

$$= \frac{1}{2}\sum_{r,s} \not\epsilon_{ef} \not k'_{fg} \not\epsilon'_{gh}\, u_r(p)_h\, u_r^*(p)_d \not\epsilon^\dagger_{dc} \not k^\dagger_{cb} \not\epsilon'^\dagger_{ba}\, \bar{u}_s^\dagger(p')_a\, \bar{u}_s(p')_e$$

$$= \frac{1}{2}\sum_{r,s} \not\epsilon_{ef} \not k'_{fg} \not\epsilon'_{gh}\, u_r(p)_h\, \bar{u}_r(p)_d \not\epsilon_{dc} \not k_{cb} \not\epsilon'_{ba}\, u_s(p')_a\, \bar{u}_s(p')_e$$

$$= \frac{1}{8E_pE_{p'}}\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' (\not p + m) \not\epsilon \not k \not\epsilon' (\not p' + m)\right] \qquad p' = p + k - k'$$

$$= \frac{1}{8E_pE_{p'}}\left\{\underset{Ⓐ}{\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not\epsilon \not k \not\epsilon' \not p'\right]} + m\,\underset{Ⓑ}{\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not\epsilon \not k \not\epsilon'\right]} + m\,\underset{Ⓒ}{\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not\epsilon \not k \not\epsilon' \not p'\right]} + m^2\,\underset{Ⓓ}{\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not\epsilon \not k \not\epsilon'\right]}\right\}$$

~~Ⓐ $= \mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not\epsilon \not k \not\epsilon' (\not p + \not k - \not k')\right]$~~

~~$= \mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not\epsilon \not k \not\epsilon' \not p\right] + \mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not\epsilon \not k \not\epsilon' \not k\right] - \mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not\epsilon \not k \not\epsilon' \not k'\right]$~~

~~$= -\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not k \not\epsilon \not\epsilon' \not p\right] + 2(k\cdot\epsilon')\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not\epsilon \not k\right] - \mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not\epsilon \underset{0}{\underbrace{\not k \not k}} \not\epsilon'\right]$~~

~~$-2(k'\cdot\epsilon)\mathrm{Tr}\left[\not\epsilon' \not p \not\epsilon \not k \not\epsilon' \not k'\right] + \mathrm{Tr}\left[\underset{0}{\underbrace{\not k' \not\epsilon \not\epsilon' \not p \not\epsilon \not k \not\epsilon' \not k'}}\right]$~~

~~$= -2(\epsilon\cdot\epsilon')\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not k \not p\right] + \mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not k \not\epsilon' \not\epsilon \not p\right] - 2(k\cdot\epsilon')\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not k \not\epsilon\right]$~~

~~$-2(k'\cdot\epsilon)\mathrm{Tr}\left[\not p \not\epsilon \not k \not k'\right]$~~

~~$= 2(\epsilon\cdot\epsilon')\mathrm{Tr}\left[\not\epsilon \not\epsilon' \not k' \not p \not k \not p\right] + 2(p\cdot\epsilon)\mathrm{Tr}\left[\not\epsilon \not k' \not\epsilon' \not p \not k \not\epsilon'\right] + \mathrm{Tr}\left[\not k' \not\epsilon \not p \not k \not\epsilon' \not p\right]$~~

~~$+2(k\cdot\epsilon')\mathrm{Tr}\left[\not k' \not\epsilon' \not p \not k\right] + 2(k'\cdot\epsilon)\mathrm{Tr}\left[\not p \not k \not\epsilon \not k'\right]$~~

~~$= 4(\epsilon\cdot\epsilon')^2\mathrm{Tr}\left[\not k' \not p \not k \not p\right] - 2(\epsilon\cdot\epsilon')\mathrm{Tr}\left[\not\epsilon' \not\epsilon \not k' \not p \not k \not p\right] + 4\underset{0}{(p\cdot\epsilon)}(\epsilon\cdot\epsilon')\mathrm{Tr}\left[\not k' \not\epsilon \not p \not k\right]$~~

~~$-2\underset{0}{(p\cdot\epsilon)}\mathrm{Tr}\left[\not\epsilon' \not k' \not\epsilon' \not p \not k \not\epsilon\right] - 2(k\cdot\epsilon)\mathrm{Tr}\left[\not k' \not p \not\epsilon' \not p\right] + \mathrm{Tr}\left[\not k' \not p \not k \underset{}{\not\epsilon'} \not\epsilon' \not p\right]$~~

~~$-2(k\cdot\epsilon')\mathrm{Tr}\left[\not\epsilon' \not k' \not p \not k\right] + 2(k'\cdot\epsilon)\mathrm{Tr}\left[\not p \not k \not\epsilon \not k'\right]$~~

~~$= 8(\epsilon\cdot\epsilon')^2(p\cdot k)\mathrm{Tr}\left[\not k' \not p\right] - 4(\epsilon\cdot\epsilon')^2\mathrm{Tr}\left[\not k' \not k \underset{m^2}{\not p \not p}\right] - 4(p\cdot k)(\epsilon\cdot\epsilon')\mathrm{Tr}\left[\not\epsilon' \not\epsilon \not k' \not p\right]$~~

~~$+2(\epsilon\cdot\epsilon')\mathrm{Tr}\left[\not\epsilon' \not\epsilon \not k' \not k \not p \not p\right] + 2(k\cdot\epsilon')\mathrm{Tr}\left[\not k' \not\epsilon' \underset{m^2}{\not p \not p}\right] - \mathrm{Tr}\left[\not k' \not p \not k \not p\right]$~~

~~$-4(k\cdot\epsilon')(k\cdot\epsilon')\mathrm{Tr}\left[\not k' \not p\right] + 2(k\cdot\epsilon')\mathrm{Tr}\left[\not k \not k' \not p \not\epsilon'\right] - 2(k'\cdot\epsilon)\mathrm{Tr}\left[\not p \not\epsilon \not k \not k'\right]$~~

$p + k = p' + k'$

⇐ $k\cdot\epsilon' = p'\cdot\epsilon'$

$k'\cdot\epsilon = -p'\cdot\epsilon'$

Ⓐ $= \mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}\not{k}\not{\varepsilon}'(\not{p}+\not{k}-\not{k}')]$

$$= \mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}\not{k}\not{\varepsilon}'\not{p}] + \mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}\not{k}\not{\varepsilon}'\not{k}] - \mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}\not{k}\not{\varepsilon}'\not{k}']$$

$$= -\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{\varepsilon}\underbrace{\not{p}\not{k}}_{}\not{\varepsilon}'\not{p}] + 2(k\cdot\varepsilon')\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}\not{k}] - \mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}\not{\varepsilon}\not{\varepsilon}'\underbrace{\not{k}\not{k}}_{0}]$$

$$-2(k'\cdot\varepsilon)\mathrm{Tr}[\not{\varepsilon}'\not{p}\not{\varepsilon}\not{k}\not{\varepsilon}'\not{k}'] + \mathrm{Tr}[\underbrace{\not{k}'\not{k}'}_{0}\not{\varepsilon}\not{\varepsilon}'\not{p}\not{\varepsilon}\not{k}\not{\varepsilon}']$$

$$= -2(p\cdot k)\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{\varepsilon}\not{\varepsilon}'\not{p}] + \mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{\varepsilon}\not{k}\not{p}\not{\varepsilon}'\not{p}] - 2(k\cdot\varepsilon')\mathrm{Tr}[\not{k}\not{k}'\not{\varepsilon}'\not{p}\underbrace{\not{\varepsilon}\not{\varepsilon}}_{-1}] + 2(k'\cdot\varepsilon)\mathrm{Tr}[\not{k}'\not{p}\not{\varepsilon}\not{k}\underbrace{\not{\varepsilon}'\not{\varepsilon}'}_{-1}]$$

$$= -4(p\cdot k)(\varepsilon\cdot\varepsilon')\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}] + 2(p\cdot k)\mathrm{Tr}[\not{\varepsilon}\not{k}'\underbrace{\not{\varepsilon}'\not{\varepsilon}'}_{-1}\not{\varepsilon}\not{p}] - \mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{\varepsilon}\not{k}\not{\varepsilon}'\underbrace{\not{p}\not{p}}_{m^2}]$$

$$+2(k\cdot\varepsilon')\mathrm{Tr}[\not{k}\not{k}'\not{\varepsilon}'\not{p}] - 2(k'\cdot\varepsilon)\mathrm{Tr}[\not{k}'\not{p}\not{\varepsilon}\not{k}]$$

$$= -4(p\cdot k)(\varepsilon\cdot\varepsilon')\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{p}] - 2(p\cdot k)\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}\not{p}] - m^2\mathrm{Tr}[\not{\varepsilon}\not{k}'\not{\varepsilon}'\not{\varepsilon}\not{k}\not{\varepsilon}']$$

$$+2(k\cdot\varepsilon')\mathrm{Tr}[\not{k}\not{k}'\not{\varepsilon}'\not{p}] - 2(k'\cdot\varepsilon)\mathrm{Tr}[\not{k}'\not{p}\not{\varepsilon}\not{k}]$$

$$= -16(p\cdot k)(\varepsilon\cdot\varepsilon')\{(k'\cdot\varepsilon)\underbrace{(p\cdot\varepsilon')}_{0} - (\varepsilon\cdot\varepsilon')(p\cdot k') + \underbrace{(p\cdot\varepsilon)}_{0}(k'\cdot\varepsilon')\}$$

$$-8(p\cdot k)\{(k'\cdot\varepsilon)\underbrace{(p\cdot\varepsilon)}_{0} - (\varepsilon\cdot\varepsilon)(p\cdot k') + \underbrace{(p\cdot\varepsilon)}_{0}(k'\cdot\varepsilon)\} - m^2\mathrm{Tr}[\not{\varepsilon}\not{\varepsilon}'\not{k}'\not{k}\not{\varepsilon}\not{\varepsilon}']$$

$$+8(k\cdot\varepsilon')\{(k\cdot k')\underbrace{(p\cdot\varepsilon')}_{0} - (k\cdot\varepsilon')(p\cdot k') + (p\cdot k)\underbrace{(k'\cdot\varepsilon')}_{0}\}$$

$$-8(k'\cdot\varepsilon)\{(p\cdot k')\underbrace{(k\cdot\varepsilon)}_{0} - (k'\cdot\varepsilon)(p\cdot k) + (k\cdot k')\underbrace{(p\cdot\varepsilon)}_{0}\}$$

$$= 16(p\cdot k)(p\cdot k')(\varepsilon\cdot\varepsilon')^2 - 8(p\cdot k)(p\cdot k') - 2m^2(\varepsilon\cdot\varepsilon')\mathrm{Tr}[\not{\varepsilon}\not{\varepsilon}'\not{k}'\not{k}] + m^2\mathrm{Tr}[\not{\varepsilon}\not{\varepsilon}'\not{k}'\not{k}\not{\varepsilon}'\not{\varepsilon}]$$

$$-8(k\cdot\varepsilon')^2(p\cdot k') + 8(k'\cdot\varepsilon)^2(p\cdot k)$$

$$= 16(p\cdot k)(p\cdot k')(\varepsilon\cdot\varepsilon')^2 - 8(p\cdot k)(p\cdot k') - 8m^2(\varepsilon\cdot\varepsilon')\{(\varepsilon\cdot\varepsilon')(k\cdot k') - (k'\cdot\varepsilon)(k\cdot\varepsilon') + \underbrace{(k\cdot\varepsilon)}_{0}(k'\cdot\varepsilon')\}$$

$$-8(k\cdot\varepsilon')^2(p\cdot k') + 8(k'\cdot\varepsilon)^2(p\cdot k) + 4m^2(k\cdot k')$$

$$= 8(p\cdot k)(p\cdot k')[2(\varepsilon\cdot\varepsilon')^2 - 1] - 8(k\cdot\varepsilon')^2(p\cdot k') + 8(k'\cdot\varepsilon)^2(p\cdot k)$$

$$-8m^2(k\cdot k')(\varepsilon\cdot\varepsilon')^2 + 8m^2(\varepsilon\cdot\varepsilon')(k'\cdot\varepsilon)(k\cdot\varepsilon') + 4m^2(k\cdot k')$$

NO.

DATE

$\text{Ⓑ} + \text{Ⓒ} = m\,\mathrm{Tr}[\not\epsilon\not k'\not\epsilon'\not p\not\epsilon\not k\not\epsilon'] \cancel{+ m\,\mathrm{Tr}[\not\epsilon\not k'\not\epsilon'\not\epsilon\not k\not\epsilon'\not p] + m\,\mathrm{Tr}[\not\epsilon\not k'\not\epsilon'\not\epsilon\not k\not\epsilon'\not k]} - m\,\mathrm{Tr}[\not\epsilon\not k'\not\epsilon'\not\epsilon\not k\not\epsilon'\not k']$

$= m\,\mathrm{Tr}[\not\epsilon\not k'\not\epsilon'\not\epsilon\not p\not k\not\epsilon'] \cancel{+ m\,\mathrm{Tr}[\not\epsilon\not k'\not\epsilon'\not\epsilon\not k\not\epsilon'\not p]} + 2m(k\cdot\epsilon')\mathrm{Tr}[\not\epsilon\not k'\not\epsilon'\not\epsilon\not k] - m\,\mathrm{Tr}[\not\epsilon\not k'\not\epsilon'\not\epsilon\underset{0}{\underbrace{\not k\not k}}\not\epsilon']$

$\cancel{-2(k'\cdot\epsilon)m\,\mathrm{Tr}[\not\epsilon'\not\epsilon\not k\not\epsilon'\not k}$

$\therefore \mathrm{Tr}[\text{odd }\gamma] = 0.$

$\text{Ⓓ} = m^2\,\mathrm{Tr}[\not\epsilon\not k'\not\epsilon'\not\epsilon\not k\not\epsilon']$

$= 2m^2(\epsilon\cdot\epsilon')\mathrm{Tr}[\not\epsilon\not k'\not k\not\epsilon'] - m^2\,\mathrm{Tr}[\not\epsilon\not k'\not\epsilon\not\epsilon'\not k\not\epsilon']$

$= 8m^2(\epsilon\cdot\epsilon')\{(k'\cdot\epsilon)(k\cdot\epsilon') - \underset{0}{\underbrace{(k\cdot\epsilon)}}(k'\cdot\epsilon') + (\epsilon\cdot\epsilon')(k\cdot k')\}$

$- 2m^2(k'\cdot\epsilon)\mathrm{Tr}[\not\epsilon\not\epsilon'\not k\not\epsilon'] + m^2\,\mathrm{Tr}[\not k'\underset{-1}{\underbrace{\not\epsilon\not\epsilon}}\not\epsilon'\not k\not\epsilon']$

$= 8m^2(\epsilon\cdot\epsilon')(k'\cdot\epsilon)(k\cdot\epsilon') + 8m^2(\epsilon\cdot\epsilon')^2(k\cdot k') - 4m^2(k'\cdot\epsilon)(k\cdot\epsilon')\mathrm{Tr}[\not\epsilon\not\epsilon'] + 2m^2(k'\cdot\epsilon)\mathrm{Tr}[\not\epsilon\underset{-1}{\underbrace{\not\epsilon'\not\epsilon'}}\not k]$

$- m^2\,\mathrm{Tr}[\not k'\not\epsilon'\not k\not\epsilon']$

$= \underline{8m^2(\epsilon\cdot\epsilon')(k'\cdot\epsilon)(k\cdot\epsilon')} + 8m^2(\epsilon\cdot\epsilon')^2(k\cdot k') - 16m^2\underline{(k'\cdot\epsilon)(k\cdot\epsilon')(\epsilon\cdot\epsilon')} - 8m^2(k'\cdot\epsilon)\underset{0}{\underbrace{(k\cdot\epsilon)}}$

$+ m^2\,\mathrm{Tr}[\not\epsilon'\not k'\not k\not\epsilon']$

$= -8m^2(\epsilon\cdot\epsilon')(k'\cdot\epsilon)(k\cdot\epsilon') + 8m^2(\epsilon\cdot\epsilon')^2(k\cdot k') - 4m^2(k\cdot k')$

よって.

$$\frac{1}{2}\sum_{r,s} T_a^* T_b = \frac{1}{8E_pE_{p'}}\cdot\Big[8(p\cdot k)(p\cdot k')\{2(\varepsilon\cdot\varepsilon')^2-1\} - 8(k\cdot\varepsilon')^2(p\cdot k') + 8(k'\cdot\varepsilon)^2(p\cdot k)$$

$$\cdots - 8m^2\cancel{(k\cdot k')}(\varepsilon\cdot\varepsilon')^2 + 8m^2\cancel{(\varepsilon\cdot\varepsilon')}(k'\cdot\varepsilon)(k\cdot\varepsilon') + 4m^2\cancel{(k\cdot k')}$$

$$- 8m^2(\varepsilon\cdot\varepsilon')\cancel{(k'\cdot\varepsilon)}(k\cdot\varepsilon') + 8m^2\cancel{(\varepsilon\cdot\varepsilon')^2}(k\cdot k') - 4m^2\cancel{(k\cdot k')}\Big]$$

$$= \frac{(p\cdot k)(p\cdot k')\{2(\varepsilon\cdot\varepsilon')^2-1\} - (k\cdot\varepsilon')^2(p\cdot k') + (k'\cdot\varepsilon)^2(p\cdot k)}{E_pE_{p'}}$$

$$\therefore \frac{1}{2}\sum_{r,s}|\langle f|S|i\rangle|^2 = \frac{e^4}{4V^2\omega_k\omega_{k'}}\Bigg[\frac{2\cancel{(k\cdot\varepsilon')^2}+(p\cdot k')}{4E_pE_{p'}(p\cdot k)} + \frac{-2\cancel{(k'\cdot\varepsilon)^2}+(p\cdot k)}{4E_pE_{p'}(p\cdot k')}$$

$$+ \frac{2\{2(\varepsilon\cdot\varepsilon')^2-1\}}{4E_pE_{p'}} + \frac{-2\cancel{(k\cdot\varepsilon')^2}}{4E_pE_{p'}(p\cdot k)} + \frac{2\cancel{(k'\cdot\varepsilon)^2}}{4E_pE_{p'}(p\cdot k')}\Bigg]$$

$$\times (2\pi)^8\left[\delta^{(4)}(p+k-p'-k')\right]^2$$

$$= \frac{e^4}{4V^2\omega_k\omega_{k'}}\left[\frac{\cancel{m}\omega_{k'}}{4E_pE_{p'}\cancel{m}\omega_k} + \frac{\cancel{m}\omega_k}{4E_pE_{p'}\cancel{m}\omega_{k'}} + \frac{2\{2(\varepsilon\cdot\varepsilon')^2-1\}}{4E_pE_{p'}}\right]\times(2\pi)^8\left[\delta^{(4)}(p+k-p'-k')\right]^2$$

$$= \frac{e^4}{16V^2\omega_k\omega_{k'}E_pE_{p'}}\left[\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} + 2\{2(\varepsilon\cdot\varepsilon')^2-1\}\right]\times(2\pi)^4\cdot\lim_{T,V\to\infty} TV\delta^{(4)}(p+k-p'-k')$$

$$= \lim_{T,V\to\infty}(2\pi)^4 TV\delta^{(4)}(p+k-p'-k')\times\frac{e^4}{16V^2\omega_k\omega_{k'}E_pE_{p'}}\left[\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} + 4(\varepsilon\cdot\varepsilon')^2 - 2\right]$$

これより、

$$w = (2\pi)^4\delta^{(4)}(p+k-p'-k')\times\frac{e^4}{16V\omega_k\omega_{k'}E_pE_{p'}}\left[\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} + 4(\varepsilon\cdot\varepsilon')^2 - 2\right]$$

終状態について和をとると

$$\int d\sigma = w\frac{V}{v_{rel}}\prod\int\frac{d^3\mathbb{P}_f}{(2\pi)^3}$$

$$=\int\frac{d^3\mathbb{k}'}{(2\pi)^3}\int\frac{d^3P'}{(2\pi)^3}\frac{Ve^4}{16V\,v_{rel}\,\omega_k\omega_{k'}E_pE_{p'}}\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')-2\right]\times(2\pi)^4\delta^{(4)}(p+k-p'-k')$$

$$=\int\frac{d^3\mathbb{k}'}{(2\pi)^3}\int\frac{d^3\mathbb{P}'}{(2\pi)^3}\frac{e^4}{16\sqrt{(k\cdot p)^2-m_k^2m_p^2}\;\omega_{k'}E_{p'}}\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')-2\right]\times(2\pi)^4\delta^{(3)}(\mathbb{k}-\mathbb{P}'-\mathbb{k}')$$

$$\times\delta(m+\omega_k-E_{p'}-\omega_{k'})$$

$$=\int d^3\mathbb{k}'\frac{e^4}{64\pi^2\,m\,\omega_k\omega_{k'}E_{p'}}\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')-2\right]\times\delta(m+\omega_k-E_{p'}-\omega_{k'})$$

$$=\int_0^\infty d|\mathbb{k}'|\,|\mathbb{k}'|^2\int d\Omega\,\frac{\alpha^2}{4m\,\omega_k\omega_{k'}E_{p'}}\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')-2\right]\delta(m+\omega_k-E_{p'}-\omega_{k'})$$

ここで，

$$\frac{\partial E_{p'}}{\partial\omega_{k'}}=\frac{\omega_{k'}-\omega_k\cos\theta}{E_{p'}},\quad\frac{\partial\omega_{k'}}{\partial\omega_{k'}}=1$$ より，

$$\frac{\partial(E_{p'}+\omega_{k'})}{\partial\omega_{k'}}=\frac{E_{p'}+\omega_{k'}-\omega_k\cos\theta}{E_{p'}}=\frac{m+\omega_k-\omega_k\cos\theta}{E_{p'}}$$

$$=\frac{m+\omega_k(1-\cos\theta)}{E_{p'}}=\frac{m\omega_k}{E_{p'}\omega_{k'}}$$

$$E_{p'}^2=(\mathbb{k}-\mathbb{k}')^2+m^2$$
$$=\mathbb{k}^2+\mathbb{k}'^2-2\mathbb{k}\cdot\mathbb{k}'+m^2$$
$$=\omega_k^2+\omega_{k'}^2-2\omega_k\omega_{k'}\cos\theta+m^2.$$

- $\omega_{k'}=\dfrac{m\omega_k}{m+\omega_k(1-\cos\theta)}$
- エネルギー保存則 $m+\omega_k=E_{p'}+\omega_{k'}$

よって，

$$\int d\sigma=\int_0^\infty d\omega_{k'}\,\omega_{k'}^2\int d\Omega\,\frac{\alpha^2}{4m\,\omega_k\omega_{k'}E_{p'}}\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')-2\right]\delta(m+\omega_k-E_{p'}-\omega_{k'})$$

$$=\int_0^\infty d(E_{p'}+\omega_{k'})\frac{\partial\omega_{k'}}{\partial(E_{p'}+\omega_{k'})}\,\omega_{k'}^2\int d\Omega\,\frac{\alpha^2}{4m\,\omega_k\omega_{k'}E_{p'}}\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')-2\right]\delta(m+\omega_k-E_{p'}-\omega_{k'})$$

$$=\int d\Omega\,\frac{\alpha^2\omega_{k'}^2}{4m^2\omega_k^2}\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')-2\right]$$

$$\therefore\left(\frac{d\sigma}{d\Omega}\right)_{Lab,pol}=\frac{\alpha^2}{4m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}+4(\varepsilon\cdot\varepsilon')-2\right]$$ Klein-Nishina formula

偏極和について.

$$(\varepsilon\cdot\varepsilon')^2=(\varepsilon\cdot\varepsilon')^2=\varepsilon_i\varepsilon_j\,\varepsilon'_i\varepsilon'_j$$ より.

$$\sum_{\lambda}\varepsilon_i(\lambda)\,\varepsilon_j(\lambda)=\delta_{ij}-\frac{k_ik_j}{\mathbb{k}^2}$$

$$\frac{1}{2}\sum_{\rm Pol}(\varepsilon\cdot\varepsilon')^2=\frac{1}{2}\cdot\left(\delta_{ij}-\frac{k_ik_j}{\mathbb{k}^2}\right)\left(\delta_{ij}-\frac{k'_ik'_j}{\mathbb{k}'^2}\right)$$

$$=\frac{1}{2}\left[3+\frac{(\mathbb{k}\cdot\mathbb{k}')^2}{|\mathbb{k}|^2|\mathbb{k}'|^2}-2\right]=\frac{1}{2}\left(1+\cos^2\theta\right)$$

したがって.

$$\omega_{k'}=\frac{m\omega_k}{m+\omega_k(1-\cos\theta)}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm Lab}=\frac{\alpha^2}{4m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left[2\frac{\omega_{k'}}{\omega_k}+2\frac{\omega_k}{\omega_{k'}}+2\left(1+\cos^2\theta\right)-4\right]$$

$$=\frac{\alpha^2}{2m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left[\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}-\sin^2\theta\right]$$

$\omega_{k'}=\dfrac{m\omega_k}{m+\omega_k(1-\cos\theta)}$ を用いて書き換えると、

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm Lab}=\frac{\alpha^2}{2}\cdot\frac{1}{[m+\omega_k(1-\cos\theta)]^2}\left[\frac{\left(\frac{m\omega_k}{m+\omega_k(1-\cos\theta)}\right)^2+\omega_k^2}{\frac{m\omega_k^2}{m+\omega_k(1-\cos\theta)}}-\sin^2\theta\right]$$

$$=\frac{\alpha^2}{2}\cdot\frac{1}{[m+\omega_k(1-\cos\theta)]^2}\cdot\left[\frac{m^2+[m+\omega_k(1-\cos\theta)]^2}{m[m+\omega_k(1-\cos\theta)]}-\sin^2\theta\right]$$

$$=\frac{\alpha^2}{2}\,\frac{1}{[m+\omega_k(1-\cos\theta)]^2}\left[\frac{\underline{m^2+m^2}+\omega_k^2(1-\cos\theta)^2+\underline{2m\omega_k(1-\cos\theta)}}{m[m+\omega_k(1-\cos\theta)]}-\sin^2\theta\right]$$

$$=\frac{\alpha^2}{2}\,\frac{1}{[m+\omega_k(1-\cos\theta)]^2}\cdot\left[\frac{\omega_k^2(1-\cos\theta)^2}{m[m+\omega_k(1-\cos\theta)]}+1+\cos^2\theta\right]$$

$\omega_k \ll m$ 、つまり、光子のエネルギーがターゲット電子より小さいとき、

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\alpha^2}{2} \cdot \frac{1}{m^2} \cdot (1+\cos^2\theta) = \frac{\alpha^2}{2m^2}(1+\cos^2\theta)$$

$$= r_0^2 \cdot \frac{1+\cos^2\theta}{2} \qquad \underbrace{r_0 = \frac{e^2}{4\pi mc^2} = \frac{e^2}{4\pi\hbar c} \cdot \frac{\hbar}{mc}}_{\text{古典電子半径}}$$

全断面積は、

$$\sigma = \int d\Omega \, r_0^2 \frac{1+\cos^2\theta}{2} = \int_{-1}^{1} dz \int_0^{2\pi} d\varphi \, r_0^2 \frac{1+z^2}{2}$$

$$= \pi r_0^2 \cdot \left[z + \frac{z^3}{3}\right]_{-1}^{1} = \frac{8\pi}{3} r_0^2 \quad \cdots \text{Thomson の断面積}$$